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Surface Instabilities in Nonlinear Elastic Materials

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Joel Cawte

Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally.

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Joel Cawte

Summary

In this thesis, we work in the setting of nonlinear compressible and incompressible hyperelasticity.

We focus on surface instabilities, in particular sulcus (crease) formation and surface wrinkling at the free boundary of a deformed elastic body. We associate the occurrence of these phenomena with the concepts of strong and weak local minimisers in the calculus of variations.

We study conditions under which crease formation is energetically favourable, and give new algebraic criteria for the initiation of surface wrinkling by studying solutions of the underlying linearised equilibrium equations.

This study was originally initiated in order to understand the failure (via creasing) of a rubber component used in oil exploration equipment.

Acknowledgements

Undeniably the first person I should thank is my supervisor Jey Sivaloganathan. I will forever be in his debt for the numerous stimulating conversations we have had, the mathematical guidance he has offered, and his unbelievable patience in the writing up process.

I have thoroughly enjoyed being a part of the mathematics department in Bath. The level of togetherness in the PhD community has been incredibly helpful in keeping me grounded. I have treasured the daily company of open-eared fellow students, of whom I am very grateful to have met.

I am particularly grateful for being welcomed into the 'Wednesday group', in which I have met friends for life. Daniel Green, Leonard Hardiman, James Green, Kieran Jarrett, Matt Thomas, Alice Davies, Bec Green, and Jack Betteridge, I have loved every minute spent playing games and watching ridiculous films with you. Particularly fond memories of mine will always be playing Super Smash Bros. Melee for the Nintendo Gamecube late into the night.

The support and love of my family all the way from home in Maidenhead has been a solid foundation, upon which I have remained strong enough to write this thesis and stay committed to it. They have always been there for me in my numerous times of need, and this thesis would not exist without them.

A special acknowledgement is deserved for my wonderful partner Beth Barry, of whom I have never met a more supportive, kind, and loving person. She has kept me sane through this process, and has never let me down.

Lastly, I thank God for the opportunity to express myself in this thesis, and for giving me the strength to complete it.

Contents

Chapter 1

Introduction

1.1 Overview

Consider the free surface of a block of elastic material modelled as occupying a half-space, subject to compression parallel to its free surface. As the compression is increased, the surface of the block may deform homogeneously, or instabilities may develop at the free surface. Two such possible instabilities are *surface wrinkling*, where small, stationary, sinusoidal waves appear, and *surface creasing*, where a sharp, self-contacting fold occurs at (possibly more than) one point on the surface, as depicted in Figure 1-1.

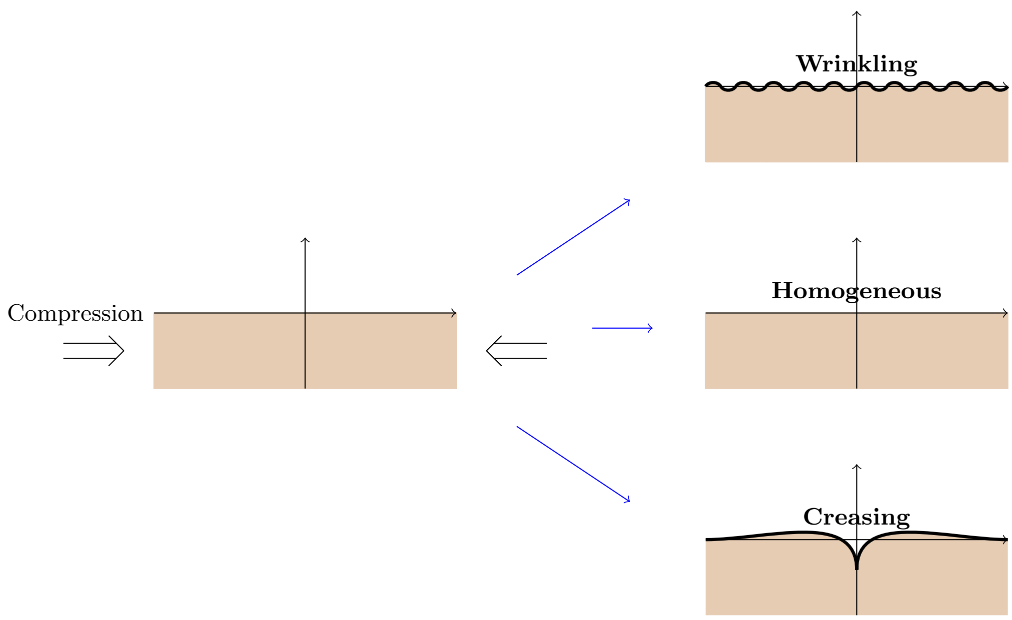


Figure 1-1

Weatherford International, an Oil and Gas firm, encountered problems related to surface creasing in their Rotary Steerable System (RSS) which is designed for drilling below the earth's surface. The design includes a synthetic rubber diaphragm in the shape of a short, slightly tapered tube (see figure 1-2). The diaphragm allows a balance of pressure between the internal hydraulic oil and the external unwanted fluids and debris by allowing a volume change in the oil, in response to the large pressures and temperatures encountered several kilometres underground.[1] After use, as the diaphragm returned to its original volume, creases would sometimes develop on the inner surface of the diaphragm parallel to the axis of symmetry, due to compressive strains on the inner surface (see figure 1-3).

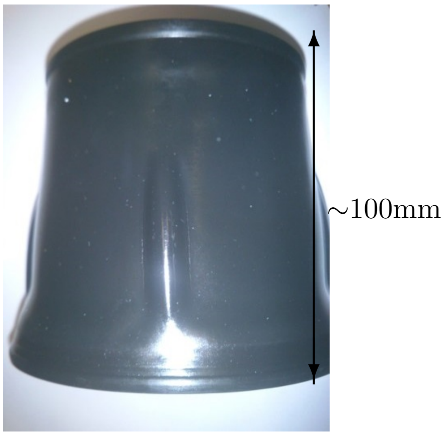


Figure 1-2: A photo of the rubber diaphragm (source: [Kuk14]). Note that the thin white strip in the centre is a refraction of light on a ridge of the smooth outer surface, not a crease.

Figure 1-3: A sulcus on the inner surface of the rubber diaphragm (source: project sponsor via Dr P. Kukian, private communication)

One of the first mathematical studies of surface wrinkling was by Biot in [Bio63], where it is shown that an incompressible, neo-Hookean material admits "instabilities" in the form of surface wrinkles at a critical compression, specifically when the compressive

[1]At these depths, temperatures can reach 180°C, and quasi-static pressures can reach 1800 bar.

strain reaches the approximate value

$$\lambda_1 = 0.544.$$

Biot does this by considering an incompressible, neo-Hookean material occupying the two-dimensional half-space

$$\{(x_1, x_2) \in \mathbb{R} \times (-\infty, 0)\}$$

in its reference configuration, and subject to compression in the x_1-direction. There is a corresponding homogeneous state satisfying the equilibrium equations in which the free surface remains planar. As the degree of compression increases, there is a critical compression at which the equilibrium equations, linearised around the homogeneous equilibrium solution, possess non-trivial solutions corresponding to stationary surface waves of arbitrary period. Biot implicitly associates this with the existence of a bifurcating branch of "wrinkling solutions" to the full equilibrium equations.[2]

Many other researchers have mistaken Biot's work in finding nontrivial solutions to the linearised problem as a proof for existence of a bifurcating branch leading to a loss of stability. On the contrary, this approach currently lacks a rigorous foundation. See Healey and Montes-Pizarro [HMP03], Simpson and Spector [SS08a], or Negrón-Marrero and Montes-Pizarro [NMMP12], for examples in nonlinear elasticity where a rigorous bifurcation analysis is carried out to prove the existence of bifurcating branches from a critical compression.

Up until Simpson and Spector's work in [SS87], it was not well understood how a boundary value problem on a bounded domain is related to its corresponding half-space problem. In [SS87], the complementing condition and Agmon's condition [ADN59] are shown to play an important role in the study of weak local minimisers (i.e. a local minimiser in the C^1 topology) of the global problem. Furthermore, failure of the complementing condition is in fact identical to the existence of nontrivial solutions to the aforementioned half-space problem.[3]

The known results relating weak local minimisers to the half-space problem in the context of *incompressible* elasticity are not as complete as in the compressible case. Based on the variational work of Fosdick and MacSithigh [FM86], it has been shown by MacSithigh in [Mac05] that a weak local minimiser necessarily satisfies (an

[2]For examples of half-space problems in elasticity, see [Tho69], [UB74], [SS08a], and [NMMP12] for half-space problems in compressible elasticity, and [Bio63], [Now69], [DO90], and [CYW18] for those in the incompressible setting.

[3]See [SS87], [SS89], [MS98], and [NMMP11] for examples of the complementing condition and Agmon's condition in nonlinear elasticity.

incompressible version of) Agmon's condition, which corresponds to another half-space problem similar to that of Biot, Nowinski, and Chen et al cited above.

The problem of surface creasing has proved to be mathematically less tractable than surface wrinkling. Gent and Cho [GC99] provide compelling experimental evidence with rubber elastomers that surface creasing appears to occur at a lower compression than surface wrinkling, estimating the critical strain to be

$$\lambda_1 = 0.65 \pm 0.07.$$

Gent and Cho remark that the cause of the discrepancy between their results and Biot's 'wrinkling' instability criterion are unknown. Some other early studies (for example, [CD08]) incorrectly claim that Biot's critical compression for the onset of wrinkling also predicted the onset of surface creasing, despite the disparity in their predicted critical compression. However, Trujillo et al [TKH08] provide experimental evidence for the claim that, at the onset of surface instability, the observed creases appear to form from the nucleation of infinitesimally small sharp folds, and not from wrinkling modes. On the other hand, Cao and Hutchinson in [CH12] argue that wrinkling is "highly imperfection sensitive", and that even if a wrinkle instability forms, it quickly develops into a crease.

Many different attempts have been made at forming a mathematical understanding of Gent and Cho's findings. One of the early successful studies is a finite element analysis by Hong et al in [HZS09], which shows that the incompressible neo-Hookean stored energy of a two-dimensional large block (approximating a half-space) in a compressed, creased state (enforcing self contact) is lower than that of a comparable homogeneous state at a compression $\lambda_1 \approx 0.65$, in approximate agreement with Gent and Cho's experiments. Hohlfeld and Mahadevan in [HM11] simulate the bending of a two-dimensional strip via a finite element analysis. They seek extrema of the incompressible neo-Hookean stored energy with the addition of a small surface energy term, and explore the limit as this term approaches zero. They find that the strain at the 'folding point' depicted in Figure 1-4 approaches $\lambda_1 \approx 0.646$ as the surface energy tends to zero, consistent with Gent and Cho's estimate for creasing. Matched asymptotics have only fairly recently been applied to crease formation by Ciarletta in [Cia18]. Although the study claims to present a theoretical explanation for the critical compression for creasing at $\lambda_1 \approx 0.638$, aspects of the derivation of this result were questioned by peer reviewers and remain controversial.

A key observation first made by Hohlfeld and Mahadevan in [HM11] is that surface creasing is a different type of instability to surface wrinkling. We see this by considering

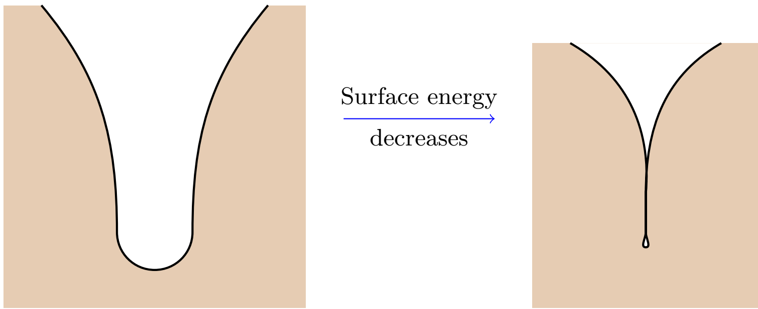


Figure 1-4: A close-up of the strip where the sulcus forms. As the surface energy decreases, extrema of the total energy tend towards a crease. [HM11]

the transition to either the wrinkled surface or the creased surface depicted in Figure 1-1. An infinitesimal, wave-like bifurcation will be C^1-close to the homogeneous state, while an infinitesimal, crease-like bifurcation will only be uniformly close. Surface creasing therefore (at least seemingly) appears to correspond to a situation in which a weak local minimiser (i.e. the homogeneous equilibrium state) fails to also be a *strong* local minimiser (i.e. a local minimiser in the C^0 topology). Two well known necessary conditions for strong local minimality are quasiconvexity [Mor52], and quasiconvexity at the boundary [BM84]. However, to date there are no known general methods for verifying quasiconvexity or quasiconvexity at the boundary, especially in the context of crease nucleation. Consequently, neither seem to have yet been implemented as possible tools to theoretically explain the onset for crease nucleation.

1.2 Preliminaries and notation

The Einstein summation convention (summing over repeated indices) will be assumed for this thesis, except in Chapters 2 and 3.

1.2.1 Deformations

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, with piecewise smooth boundary $\partial\Omega$, denote the region occupied by an elastic body in its reference configuration. Let $\partial\Omega = \partial\Omega_D \cup \partial\Omega_T \cup \partial\Omega_S$ be a disjoint partition of the boundary. We allow the possibilities $\partial\Omega = \partial\Omega_D$, $\partial\Omega = \partial\Omega_T$, or $\partial\Omega = \partial\Omega_S$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$ denote cartesian coordinates in the reference configuration. A *deformation* of the body is an invertible map $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^n$, which maps points $\mathbf{x} \in \Omega$ in the reference configuration to $\boldsymbol{\varphi}(\mathbf{x}) = (\varphi_1(\mathbf{x}), \ldots, \varphi_n(\mathbf{x}))$ in the deformed configuration.[4] We associate the deformation gradient of $\boldsymbol{\varphi}$ at $\mathbf{x}$, denoted by

[4]Coordinates in the reference and deformed configuration are sometimes referred to as material (or Lagrangian) and spatial (or Eulerian) coordinates, respectively.

$\nabla\boldsymbol{\varphi}(\mathbf{x})$, with the $n \times n$ matrix $(\nabla\boldsymbol{\varphi}(\mathbf{x}))_{i\alpha} = \frac{\partial\varphi_i(\mathbf{x})}{\partial x_\alpha}$, for $i, \alpha = 1, \ldots, n$. We require that admissible deformations satisfy the local invertibility condition

$$\det(\nabla\boldsymbol{\varphi}(\mathbf{x})) > 0, \quad \text{for all } \mathbf{x} \in \Omega, \tag{1.1}$$

and so $\nabla\boldsymbol{\varphi}(\mathbf{x}) \in M_+^{n\times n}$ for all $\mathbf{x} \in \Omega$, where $M_+^{n\times n}$ is the set of real $n \times n$ matrices with positive determinant. For C^1 maps, this condition implies local invertibility of $\boldsymbol{\varphi}$ by the Inverse Function Theorem, and hence prevents local interpenetration of matter under such a deformation. For incompressible materials, admissible deformations are subject to the constraint

$$\det(\nabla\boldsymbol{\varphi}(\mathbf{x})) = 1, \quad \text{for all } \mathbf{x} \in \Omega, \tag{1.2}$$

in which case, $\nabla\boldsymbol{\varphi}(\mathbf{x}) \in M_1^{n\times n}$ for all $\mathbf{x} \in \Omega$, where $M_1^{n\times n}$ is the set of real $n \times n$ matrices with determinant 1.

1.2.2 Polar decomposition of $n \times n$ matrices

Theorem 1.2.1 (Polar Decomposition[5])**.** *For any* $\mathbf{F} \in M_+^{n\times n}$*, there exists* $\mathbf{R} \in SO(n)$*, and symmetric positive definite matrices* $\mathbf{U}$ *and* $\mathbf{V}$*, called the 'right' and 'left stretch tensors' respectively, such that*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}.$$

This is called the polar decomposition *of* $\mathbf{F}$*, and it is unique.*

Note that $\mathbf{U}$ is the (unique) symmetric positive definite matrix such that $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$, and $\mathbf{V}$ is the (unique) symmetric positive definite matrix such that $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$, so we write $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$, and $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}$. Since any symmetric positive definite matrix has positive eigenvalues, and can be diagonalised, there exists $\mathbf{Q} \in SO(n)$ such that

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} = \mathbf{Q}\,\mathrm{diag}(v_1, \ldots, v_n)\mathbf{Q}^T, \tag{1.3}$$

where $\mathrm{diag}(v_1, \ldots, v_n)$ is the diagonal matrix with entries v_i for $i = 1, \ldots, n$, and v_i, $i = 1, \ldots, n$, are the eigenvalues of $\mathbf{U}$. It can be shown, with the aid of Theorem 1.2.1, that the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$ coincide.

Definition 1.2.2 (Principal stretches)**.** Let $\mathbf{F} = \nabla\boldsymbol{\varphi}$, where $\boldsymbol{\varphi}$ is a deformation. Then the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$ (and equivalently, $\sqrt{\mathbf{F}\mathbf{F}^T}$) given by (1.3) are called the *principal stretches* associated to $\boldsymbol{\varphi}$.

[5] See, for example, Ciarlet [Cia88].

It may also be shown, as a consequence of Theorem 1.2.1, that the characteristic polynomials for $\mathbf{U}$ and $\mathbf{V}$ coincide. Similarly, the characteristic polynomials for $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ and $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$ also coincide. The *principal invariants* of $\mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ (equivalently $\mathbf{F}\mathbf{F}^T = \mathbf{V}^2$) are given by the coefficients of the characteristic polynomial associated to $\mathbf{F}^T\mathbf{F}$ (equivalently $\mathbf{F}\mathbf{F}^T$). Namely,

$$\begin{aligned}
&I_1 = \operatorname{tr}(\mathbf{F}^T\mathbf{F}) = v_1^2 + v_2^2, \qquad I_2 = \det(\mathbf{F}^T\mathbf{F}) = (v_1v_2)^2, && \text{if } n = 2;\\
&I_1 = \operatorname{tr}(\mathbf{F}^T\mathbf{F}) = v_1^2 + v_2^2 + v_3^2, &&\\
&I_2 = \operatorname{tr}(\operatorname{Cof}(\mathbf{F}^T\mathbf{F})) = (v_1v_2)^2 + (v_1v_3)^2 + (v_2v_3)^2, &&\\
&I_3 = \det(\mathbf{F}^T\mathbf{F}) = (v_1v_2v_3)^2, && \text{if } n = 3,
\end{aligned} \tag{1.4}$$

where v_i, for $i = 1, \ldots, n$, are the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$.

1.2.3 Hyperelasticity

Compressible hyperelasticity

A material is said to be hyperelastic if there exists a function $W : \Omega \times M_+^{n\times n} \to \mathbb{R}$ such that for any deformation $\boldsymbol{\varphi}$ satisfying (1.1), the stored energy of the deformed body is given by

$$E[\boldsymbol{\varphi}] = \int_\Omega W(\mathbf{x}, \nabla\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}\mathbf{x}.$$

The function W is called the *stored energy function* of the material, and characterises the material response.

We say W is

1. *homogeneous* if

$$W(\mathbf{x}, \mathbf{F}) = W(\mathbf{F}), \quad \text{for all } \mathbf{x} \in \Omega,\ \mathbf{F} \in M_+^{n\times n}, \tag{1.5}$$

2. *frame indifferent* if

$$W(\mathbf{x}, \mathbf{Q}\mathbf{F}) = W(\mathbf{x}, \mathbf{F}), \quad \text{for all } \mathbf{x} \in \Omega,\ \mathbf{F} \in M_+^{n\times n},\ \mathbf{Q} \in SO(n), \tag{1.6}$$

3. *isotropic* if

$$W(\mathbf{x}, \mathbf{F}\mathbf{Q}) = W(\mathbf{x}, \mathbf{F}), \quad \text{for all } \mathbf{x} \in \Omega,\ \mathbf{F} \in M_+^{n\times n},\ \mathbf{Q} \in SO(n), \tag{1.7}$$

If W satisfies (1.5), (1.6), and (1.7), then there exists a symmetric function

$\Phi : (0,\infty)^n \to \mathbb{R}$ such that

$$W(\mathbf{F}) = \Phi(v_1(\mathbf{F}), \dots, v_n(\mathbf{F})), \quad \text{for all } \mathbf{F} \in M_+^{n\times n}, \tag{1.8}$$

where $v_1(\mathbf{F}), \dots, v_n(\mathbf{F})$ are the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$ (see, for example, Ciarlet [Cia88]).

For the rest of this thesis, we will assume that W is homogeneous, frame-indifferent, and isotropic (so that (1.5), (1.6), and (1.7) are satisfied).

Incompressible hyperelasticity

In the context of incompressible elasticity, a material is said to be hyperelastic if there exists a function $W^{\text{inc}} : \Omega \times M_1^{n\times n} \to \mathbb{R}$ such that for any deformation $\boldsymbol{\varphi}$ satisfying (1.2), the stored energy of the deformed body is given by

$$E^{\text{inc}}[\boldsymbol{\varphi}] = \int_\Omega W^{\text{inc}}(\mathbf{x}, \nabla\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

where W^{inc} is the (incompressible) stored energy function of the material.

Define the set

$$\Lambda_n = \{(v_1, \dots, v_n) \in (0,\infty)^n \mid v_1 \dots v_n = 1\}. \tag{1.9}$$

We will assume for this thesis that the function W^{inc} is homogeneous, frame-indifferent, and isotropic. Hence, there exists a symmetric function $\Phi^{\text{inc}} : \Lambda_n \to \mathbb{R}$ such that

$$W^{\text{inc}}(\mathbf{F}) = \Phi^{\text{inc}}(v_1(\mathbf{F}), \dots, v_n(\mathbf{F})), \quad \text{for all } \mathbf{F} \in M_1^{n\times n}, \tag{1.10}$$

where $v_i(\mathbf{F})$, for $i = 1, \dots, n$ are the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$. Alternatively, the stored energy function can be written as

$$W^{\text{inc}}(\mathbf{F}) = \begin{cases} h^{\text{inc}}(I_1) & \text{if } n = 2, \\ h^{\text{inc}}(I_1, I_2) & \text{if } n = 3, \end{cases} \tag{1.11}$$

where I_1 and I_2 are the first and second principal invariants of $\mathbf{F}^T\mathbf{F}$, given by (1.4)[6].

Remark 1.2.3. For later use we note that for any scalar function $p : \Omega \to \mathbb{R}$, W^{inc} is indistinguishable on $M_1^{n\times n}$ from the modified incompressible stored energy function

$$\widetilde{W}^{\text{inc}}(\mathbf{x}, \mathbf{F}) = W^{\text{inc}}(\mathbf{F}) - p(\mathbf{x})(\det(\mathbf{F}) - 1),$$

[6] See the compressible analogue (1.8).

for all $\mathbf{F} \in M_1^{n\times n}$.

1.2.4 Stress tensors

Compressible elasticity

We denote by $\mathbf{S} : \Omega \to M^{n\times n}$ the (first) Piola-Kirchhoff stress tensor, and by $\widehat{\mathbf{S}}$ the response function for the Piola-Kirchhoff stress tensor, given by

$$\widehat{\mathbf{S}}(\mathbf{F}) = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}, \quad \text{for all } \mathbf{F} \in M_+^{n\times n}, \tag{1.12}$$

which obey the constitutive relation

$$\mathbf{S}(\mathbf{x}) = \widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x})), \quad \text{for all } \mathbf{x} \in \Omega. \tag{1.13}$$

We denote by $\mathbf{T} : \boldsymbol{\varphi}(\Omega) \to M^{n\times n}$ the *Cauchy Stress Tensor*, and by $\widehat{\mathbf{T}}$ the response function for the Cauchy Stress Tensor, given by

$$\widehat{\mathbf{T}}(\mathbf{F}) = \frac{1}{\det(\mathbf{F})}\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^T, \quad \text{for all } \mathbf{F} \in M_+^{n\times n}, \tag{1.14}$$

which obey the constitutive relation

$$\mathbf{T}(\mathbf{y}) = \widehat{\mathbf{T}}(\nabla\boldsymbol{\varphi}(\mathbf{x})), \quad \text{for all } \mathbf{x} \in \Omega,\ \mathbf{y} \in \boldsymbol{\varphi}(\Omega) \text{ such that } \mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}). \tag{1.15}$$

The *equilibrium equations of hyperelasticity*, under zero body forces, are the Euler-Lagrange equations corresponding to E, given by

$$\frac{\partial}{\partial x_\alpha} S_{i\alpha}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega,\ i = 1, \dots, n. \tag{1.16}$$

It is possible to write (1.16) in spatial/deformed coordinates, specifically

$$\frac{\partial}{\partial y_j} T_{ij}(\mathbf{y}) = 0, \qquad \mathbf{y} \in \boldsymbol{\varphi}(\Omega),\ i = 1, \dots, n. \tag{1.17}$$

Where no confusion arises as a result, we will not distinguish between a stress tensor and its corresponding response function.

Remark 1.2.4. Note that any *homogeneous* deformation

$$\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{A}\mathbf{x},$$

where $\mathbf{A} \in M_+^{n\times n}$ is always a solution of (1.16) and (1.17).

Incompressible elasticity

We define the Piola-Kirchhoff extra stress tensor by

$$\bar{\mathbf{S}}(\mathbf{F}) = \frac{\partial W^{\text{inc}}(\mathbf{F})}{\partial \mathbf{F}}, \quad \text{for all } \mathbf{F} \in M_1^{n\times n},$$

and the corresponding Cauchy extra stress tensor by

$$\bar{\mathbf{T}}(\mathbf{F}) = \frac{\partial W^{\text{inc}}(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^T, \quad \text{for all } \mathbf{F} \in M_1^{n\times n}.$$

The Piola-Kirchhoff stress tensor $\mathbf{S}$ obeys the constitutive relation

$$\mathbf{S}(\mathbf{x}) = -p(\mathbf{x})(\nabla\boldsymbol{\varphi}(\mathbf{x}))^{-T} + \bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x})), \quad \text{for all } \mathbf{x} \in \Omega,$$

where $p : \Omega \to \mathbb{R}$ is the pressure, which is a Lagrange multiplier corresponding to the constraint of incompressibility. The corresponding Cauchy stress tensor obeys the constitutive relation

$$\mathbf{T}(\mathbf{y}) = -\widetilde{p}(\boldsymbol{\varphi}(\mathbf{x}))\mathbb{1} + \bar{\mathbf{T}}(\nabla\boldsymbol{\varphi}(\mathbf{x})), \quad \text{for all } \mathbf{x} \in \Omega,\ \mathbf{y} \in \boldsymbol{\varphi}(\Omega) \text{ such that } \mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}),$$

where $\widetilde{p} : \boldsymbol{\varphi}(\Omega) \to \mathbb{R}$ is such that $\widetilde{p}(\boldsymbol{\varphi}(\mathbf{x})) = p(\mathbf{x})$ for all $\mathbf{x} \in \Omega$. The *equilibrium equations of incompressible hyperelasticity* under zero body force are the Euler-Lagrange equations corresponding to E^{inc}, given by

$$\frac{\partial}{\partial x_\alpha} S_{i\alpha}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega, i = 1, \ldots, n. \tag{1.18}$$

We may write (1.18) in terms of deformed coordinates by

$$\frac{\partial}{\partial y_j} T_{ij}(\mathbf{y}) = 0, \qquad \mathbf{y} \in \boldsymbol{\varphi}(\Omega),\ i = 1, \ldots, n. \tag{1.19}$$

1.2.5 Boundary value problems

Suppose that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_T \cup \partial\Omega_S$ is a partition of the boundary of Ω. A typical variational problem in homogeneous hyperelasticity is to minimise

$$E[\boldsymbol{\varphi}] = \int_\Omega W(\nabla\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}\mathbf{x} - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x})\ \mathrm{d}S(\mathbf{x}) \tag{1.20}$$

over a set of admissible deformations satisfying prescribed boundary data on $\partial\Omega_D$ and $\partial\Omega_S$, where $\mathbf{t} : \partial\Omega_T \to \mathbb{R}^n$ is a prescribed traction field corresponding to the force per unit area acting on $\partial\Omega_T$. Any sufficiently smooth minimising deformation then satisfies the equilibrium equations (1.16) and a corresponding set of boundary conditions depending on the admissible set. The following will summarise the main cases relevant to this thesis.

Displacement boundary value problems

In this case, $\partial\Omega = \partial\Omega_D$, and admissible deformations are required to satisfy the *displacement boundary condition*

$$\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega_D, \tag{1.21}$$

where $\mathbf{f} : \partial\Omega_D \to \mathbb{R}^n$ is some given, smooth map.

(Dead load) traction boundary value problems

In this case, $\partial\Omega = \partial\Omega_T$ (so that deformations are not subject to any displacement boundary data). Then

$$\widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega_T, \tag{1.22}$$

where $\mathbf{n}(\mathbf{x})$ is the unit normal to $\partial\Omega_T$ at $\mathbf{x}$, and $\mathbf{t}$ is given. The *traction boundary condition* (1.22) is in fact a natural boundary condition for the variational problem associated to E.

Slip boundary value problems

In this case, $\partial\Omega = \partial\Omega_S$, and admissible deformations are required to satisfy the *slip boundary condition*

$$\boldsymbol{\varphi}(\mathbf{x}) \in \partial Y, \qquad \mathbf{x} \in \partial\Omega_S, \tag{1.23}$$

where ∂Y is the boundary of some prescribed region $Y \subset \mathbb{R}^n$. Then we have that

$$\boldsymbol{\tau}(\boldsymbol{\varphi}(\mathbf{x}))^T\widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega_S, \tag{1.24}$$

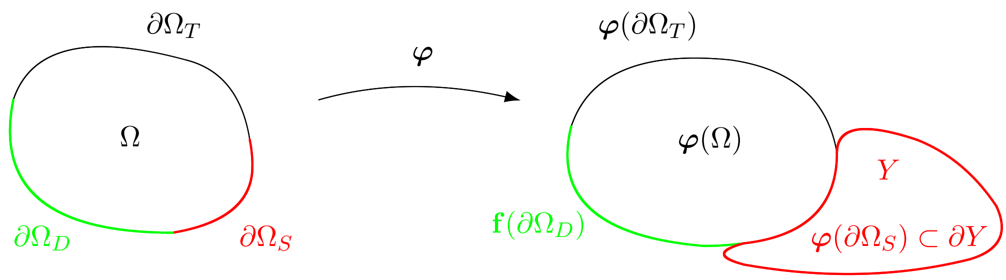
for all vectors $\boldsymbol{\tau}(\boldsymbol{\varphi}(\mathbf{x}))$ tangent to ∂Y at $\boldsymbol{\varphi}(\mathbf{x})$.[7] The additional boundary condition (1.24) arises as a natural boundary condition for the variational problem associated to E.

[7] That is, $\boldsymbol{\tau}(\boldsymbol{\varphi}(\mathbf{x})) \cdot \mathbf{n}(\boldsymbol{\varphi}(\mathbf{x})) = 0$ for all $\mathbf{x} \in \partial\Omega_S$, where $\mathbf{n}(\boldsymbol{\varphi}(\mathbf{x}))$ is the normal to $\boldsymbol{\varphi}(\partial\Omega_S)$ (and ∂Y) at $\boldsymbol{\varphi}(\mathbf{x})$ in the deformed configuration.

Mixed boundary value problems

In this case, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_T \cup \partial\Omega_S$, where $\partial\Omega_D$, $\partial\Omega_T$, and $\partial\Omega_S$ are all nonempty. Let us now require any deformation $\boldsymbol{\varphi}$ to satisfy the boundary condition (1.21) and (1.23),[8] and no boundary condition imposed on remaining part of the boundary $\partial\Omega_T$. Then, in addition to these boundary conditions, $\boldsymbol{\varphi}$ must also satisfy the natural boundary conditions (1.22) and (1.24).

Remark 1.2.5. The displacement, traction, and sliding boundary value problems, and any combination thereof, are special cases of the system (1.16) and (1.21)-(1.24) on variously taking $\partial\Omega_D$, $\partial\Omega_T$, or $\partial\Omega_S$ to be empty. For example, the case where $\partial\Omega_T = \varnothing$ and $\partial\Omega_S = \varnothing$ implies that $\partial\Omega = \partial\Omega_D$, resulting in the displacement boundary value problem where the only boundary condition is (1.21).



Incompressible mixed boundary value problems

Consider the mixed boundary value problem where deformations must satisfy (1.21) and (1.24), but now we further restrict deformations to satisfy the incompressibility constraint (1.2). Then the boundary value problem corresponding to this *incompressible* system is given by

$$\frac{\partial}{\partial x_\alpha}\bar{S}_{i\alpha}(\nabla\boldsymbol{\varphi}(\mathbf{x})) - (\mathrm{Cof}\,\nabla\boldsymbol{\varphi}(\mathbf{x}))_{i\alpha}\frac{\partial\varphi}{\partial x_\alpha}(\mathbf{x}) = 0, \qquad \mathbf{x}\in\Omega,\ i=1,\dots,n, \tag{1.25a}$$

$$\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}\in\partial\Omega_D, \tag{1.25b}$$

$$\boldsymbol{\varphi}(\mathbf{x}) \in \partial Y, \qquad \mathbf{x}\in\partial\Omega_S \tag{1.25c}$$

$$\boldsymbol{\tau}(\boldsymbol{\varphi}(\mathbf{x}))^T\left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x})) - p(\mathbf{x})\mathrm{Cof}\,\nabla\boldsymbol{\varphi}(\mathbf{x})\right)\mathbf{n}(\mathbf{x}) = 0, \qquad \mathbf{x}\in\partial\Omega_S, \tag{1.25d}$$

$$\left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x})) - p(\mathbf{x})\mathrm{Cof}\,\nabla\boldsymbol{\varphi}(\mathbf{x})\right)\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), \qquad \mathbf{x}\in\partial\Omega_T, \tag{1.25e}$$

for all vectors $\boldsymbol{\tau}(\boldsymbol{\varphi}(\mathbf{x}))$ tangent to ∂Y at $\boldsymbol{\varphi}(\mathbf{x})$, where $p : \Omega \to \mathbb{R}$ is a pressure function acting as a Lagrange multiplier corresponding to the incompressibility constraint (1.2).

[8] We assume that $\mathbf{f}\cdot\mathbf{n} = g$ and $\nabla\mathbf{f}\cdot\mathbf{n} = \nabla g$ at all points joining the segments $\partial\Omega_D$ and $\partial\Omega_S$.

In the following two subsections, for compressible and incompressible hyperelasticity respectively, we will review further necessary conditions for a deformation to be a minimiser.

1.2.6 Minimisers in compressible hyperelasticity

We are interested in minimizing the stored energy E given by (1.20), over the set of admissible deformations given by

$$\mathcal{A} = \{\boldsymbol{\varphi} \in C^1(\overline{\Omega}, \mathbb{R}^n) \mid \boldsymbol{\varphi} \text{ satisfies } (1.1), (1.21)\}, \tag{1.26}$$

in the case $\partial\Omega_S = \varnothing$ (so that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_T$), and $\partial\Omega_T \neq \varnothing$ (we allow the possibility that $\partial\Omega = \partial\Omega_T$). Define the set of *variations*

$$\mathcal{V} = \{\mathbf{u} \in C^1(\overline{\Omega}, \mathbb{R}^n) \mid \mathbf{u} = 0 \text{ on } \partial\Omega_D\}. \tag{1.27}$$

Definition 1.2.6 (Local minimisers)**.** We say the deformation $\boldsymbol{\varphi} \in \mathcal{A}$ is a *weak local minimiser* of E if there exists $\epsilon > 0$ such that $E(\boldsymbol{\varphi} + \mathbf{u}) \geq E(\boldsymbol{\varphi})$ for all $\mathbf{u} \in \mathcal{V}$ satisfying $\|\mathbf{u}\|_{1,\infty} < \epsilon$. Similarly, $\boldsymbol{\varphi} \in \mathcal{A}$ is a *strong local minimiser* of E if there exists $\epsilon > 0$ such that $E(\boldsymbol{\varphi} + \mathbf{u}) \geq E(\boldsymbol{\varphi})$ for all $\mathbf{u} \in \mathcal{V}$ satisfying $\|\mathbf{u}\|_\infty < \epsilon$.

Since $\|\mathbf{u}\|_{1,\infty} < \epsilon$ implies $\|\mathbf{u}\|_\infty < \epsilon$, any strong local minimiser is also a weak local minimiser.

Definition 1.2.7 (Elasticity tensor)**.** For a given matrix $\mathbf{F} \in M_+^{n\times n}$, we define the corresponding *elasticity tensor* $\mathbf{C}(\mathbf{F})$ by

$$\mathbf{C}(\mathbf{F}) = \frac{\partial^2 W(\mathbf{F})}{\partial \mathbf{F}^2} = \left(\frac{\partial^2 W(\mathbf{F})}{\partial F_{i\alpha}\partial F_{j\beta}}\right). \tag{1.28}$$

We denote

$$\mathbf{C}(\mathbf{F})[\mathbf{A}] = \left(\frac{\partial^2 W(\mathbf{F})}{\partial F_{i\alpha}\partial F_{j\beta}} A_{j\beta}\right),$$
$$\mathbf{C}(\mathbf{F})[\mathbf{A}, \mathbf{B}] = \frac{\partial^2 W(\mathbf{F})}{\partial F_{i\alpha}\partial F_{j\beta}} A_{i\alpha} B_{j\beta}.$$

Definition 1.2.8 (Legendre-Hadamard condition)**.** Given $\mathbf{F} \in M_+^{n\times n}$, we say the elasticity tensor $\mathbf{C}(\mathbf{F})$ satisfies the *Legendre-Hadamard condition* at $\mathbf{F}$ if

$$\mathbf{C}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}, \mathbf{a}\otimes\mathbf{b}] \geq 0, \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n. \tag{1.29}$$

Definition 1.2.9 (Strong ellipticity)**.** Given $\mathbf{F} \in M_+^{n\times n}$, we say the elasticity tensor $\mathbf{C}(\mathbf{F})$ is *strongly elliptic* at $\mathbf{F}$ if

$$\mathbf{C}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}, \mathbf{a}\otimes\mathbf{b}] > 0, \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n. \tag{1.30}$$

The second variation

Definition 1.2.10 (Second variation)**.** For the functional E given by (1.20) and a deformation $\boldsymbol{\varphi} \in \mathcal{A}$, we define the *second variation* of E at $\boldsymbol{\varphi}$ to be the quadratic functional on $\mathcal{V}$ given by

$$\begin{aligned}\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] &:= \frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} E(\boldsymbol{\varphi} + \epsilon\mathbf{u})\big|_{\epsilon=0} \\ &= \int_\Omega \mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}))[\nabla\mathbf{u}, \nabla\mathbf{u}]\ \mathrm{d}\mathbf{x}, \qquad \mathbf{u} \in \mathcal{V}. \end{aligned} \tag{1.31}$$

The Euler-Lagrange equations for the second variation (if $\boldsymbol{\varphi} \in C^2(\overline{\Omega}, \mathbb{R}^n)$) are given by

$$\frac{\partial}{\partial x_\alpha}\left(C^{ij}_{\alpha\beta}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\frac{\partial u_j}{\partial x_\beta}\right) = 0, \qquad \mathbf{x} \in \Omega,\ i = 1, \ldots, n. \tag{1.32}$$

We note that this system corresponds to the linearisation of the nonlinear system (1.16) around $\boldsymbol{\varphi}$.

Definition 1.2.11 (Nonnegativity and positivity of the second variation)**.** Given $\boldsymbol{\varphi} \in \mathcal{A}$, the second variation is said to be *nonnegative* if

$$\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] \geq 0, \quad \text{for all } \mathbf{u} \in \mathcal{V}.$$

Similarly, given $\boldsymbol{\varphi} \in \mathcal{A}$, the second variation is said to be *strictly positive* if

$$\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] > 0, \quad \text{for all } \mathbf{u} \in \mathcal{V} \text{ such that } \mathbf{u} \not\equiv 0.$$

Given $\boldsymbol{\varphi} \in \mathcal{A}$, we say the second variation is *uniformly positive* if for some $\gamma > 0$,

$$\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] \geq \gamma\|\mathbf{u}\|^2_{1,2} = \gamma\int_\Omega |\mathbf{u}|^2 + |\nabla\mathbf{u}|^2\ \mathrm{d}\mathbf{x}, \quad \text{for all } \mathbf{u} \in \mathcal{V}.$$

It is well known that if $\boldsymbol{\varphi}$ is a weak local minimiser, then $\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] \geq 0$ for all $\mathbf{u} \in \mathcal{V}$[9]. On the other hand, a sufficient condition for $\boldsymbol{\varphi}$ to be a weak local minimiser

[9] Also known as Hadamard stability, Cf. Hadamard [Had03, pg. 252]. See also Gurtin and Spector

is that the second variation is uniformly positive.

Half-space problems

Consider the boundary value problem corresponding to the Euler-Lagrange equations for the second variation (1.31), given by (1.32) along with the boundary conditions

$$\mathbf{u}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in \partial\Omega_D, \tag{1.33}$$

$$\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}))[\nabla\mathbf{u}(\mathbf{x})]\mathbf{n}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in \partial\Omega_T, \tag{1.34}$$

where $\mathbf{n}(\mathbf{x})$ is the normal to $\partial\Omega$ at $\mathbf{x}$. More information on the solvability of (1.32)-(1.34) may be obtained[10] by considering the following.

Let $\mathbf{x}_0 \in \partial\Omega_T$, and let $\mathbf{n} = \mathbf{n}(\mathbf{x}_0)$ denote the outward unit normal to $\partial\Omega_T$ at $\mathbf{x}_0$. Let $\alpha \geq 0$, and consider the auxiliary problem on the half space

$$H_{\mathbf{n}} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{n} < 0\},$$

where we seek solutions $\mathbf{v} : H_{\mathbf{n}} \to \mathbb{R}^n$ of the boundary value problem

$$\operatorname{div}\left(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\nabla\mathbf{v}]\right) = \alpha^2\mathbf{v}, \qquad \mathbf{x} \in H_{\mathbf{n}}, \tag{1.35a}$$

$$\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\nabla\mathbf{v}]\mathbf{n} = 0, \qquad \mathbf{x} \in \partial H_{\mathbf{n}}. \tag{1.35b}$$

See Figure 1-5. We consider solutions of (1.35) of the form

$$\mathbf{v}(\mathbf{x}) = \operatorname{Re}(\mathbf{z}(-\mathbf{n}\cdot\mathbf{x})e^{\mathrm{i}\boldsymbol{\tau}\cdot\mathbf{x}}), \tag{1.36}$$

where $\mathbf{z} : [0, \infty) \to \mathbb{C}^n$.

Definition 1.2.12 (The complementing condition)**.** Let $\alpha = 0$. We say that the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)), \mathbf{n})$ satisfies the *complementing condition* [ADN59] if for any $\boldsymbol{\tau} \in \mathbb{R}^n$ orthogonal to $\mathbf{n}$, the only solution of (1.35) of the form (1.36) that decays to zero as $\mathbf{n}\cdot\mathbf{x} \to -\infty$ is $\mathbf{v} \equiv \mathbf{0}$.

Definition 1.2.13 (Agmon's condition)**.** Let $\alpha > 0$. We say that the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)), \mathbf{n})$ satisfies *Agmon's condition* [ADN59] if for any $\boldsymbol{\tau} \in \mathbb{R}^n$ orthogonal to $\mathbf{n}$, the only solution of (1.35) of the form (1.36) that decays to zero as $\mathbf{n}\cdot\mathbf{x} \to -\infty$ is $\mathbf{v} \equiv \mathbf{0}$.

[GS79], Spector [Spe82], [SS87], and the references therein for further discussion of Hadamard stability and the second variation.

[10] See Theorem 1.2.16, for example.

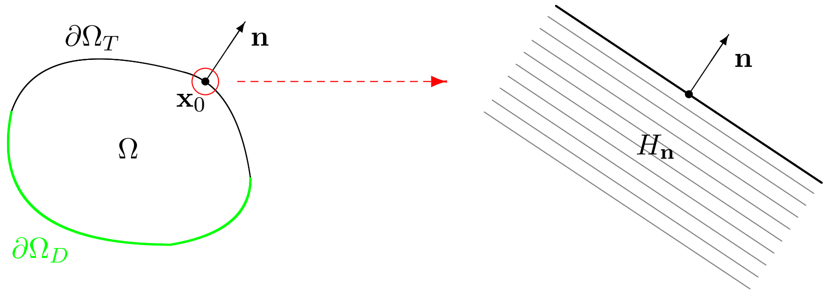


Figure 1-5: The auxiliary problem (1.35) with $\alpha = 0$ can be viewed as a rescaled limit of the boundary value problem (1.32)-(1.34) centred at $\mathbf{x}_0 \in \partial\Omega_T$.

Definition 1.2.14 (The strong complementing condition)**.** We say that the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)), \mathbf{n})$ satisfies the *strong complementing condition* if both the complementing condition and Agmon's condition hold.

Remark 1.2.15. The above conditions, stated in terms of systems of ordinary differential equations, are actually algebraic conditions on the elasticity tensor $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))$ at $\mathbf{x}_0 \in \partial\Omega_T$, which in turn depends on the stored energy function W, the deformation $\boldsymbol{\varphi}$, and the point $\mathbf{x}_0 \in \partial\Omega_T$. The solutions we seek of the form (1.36) are, by inspection, wave-like in the directions tangent to the surface, and exponentially decaying in the inward normal direction.

Theorem 1.2.16 (Simpson and Spector [SS87, Theorem 1])**.** *Let $\boldsymbol{\varphi} \in \mathcal{A}$. Suppose that the second variation $\delta^2 E(\boldsymbol{\varphi})$, given by* (1.31)*, is strictly positive. Then necessary and sufficient conditions for $\delta^2 E(\boldsymbol{\varphi})$ to be uniformly positive are:*

1. *that the elasticity tensor $\mathbf{C}(\mathbf{F})$ is strongly elliptic at $\mathbf{F} = \nabla\boldsymbol{\varphi}(\mathbf{x})$ for all $\mathbf{x} \in \Omega$; and*

2. *that for every $\mathbf{x}_0 \in \partial\Omega_T$, the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)), \mathbf{n})$ satisfies the complementing condition.*

Theorem 1.2.16 is one of many results relating the second variation to the complementing condition, see [SS87] and [SS89] for further details. Some examples of the complementing condition and Agmon's condition will be discussed in Chapter 2. For a general discussion and examples of the complementing condition in elliptic problems (including nonlinear elasticity), see Negrón-Marrero and Montes-Pizarro [NMMP11].

Quasiconvexity

We now consider necessary conditions for strong local minimisers.

Definition 1.2.17 (Quasiconvexity)**.** The stored energy function W is *quasiconvex* at $\mathbf{F}_0 \in M_+^{n\times n}$ (see Morrey [Mor52]) if

$$\begin{aligned}\int_D W(\mathbf{F}_0 + \nabla\boldsymbol{\psi}(\mathbf{x}))\ \mathrm{d}\mathbf{x} &\geq \int_D W(\mathbf{F}_0)\ \mathrm{d}\mathbf{x} \\ &= |D|W(\mathbf{F}_0)\end{aligned}$$

for any bounded domain $D \subset \mathbb{R}^n$, and for every $\boldsymbol{\psi} \in C_0^1(D, \mathbb{R}^n)$.

Remark 1.2.18. A scaling argument shows that the quasiconvexity condition is independent of the choice of domain D. Thus, if quasiconvexity holds for one such D, then it holds for all bounded domains D.

Remark 1.2.19. The quasiconvexity condition is equivalent to requiring that the homogeneous deformation $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{F}_0\mathbf{x}$ is a global energy minimiser for the corresponding pure displacement problem on D among C^1 deformations of the form $\widetilde{\boldsymbol{\varphi}}(\mathbf{x}) = \mathbf{F}_0\mathbf{x} + \psi(\mathbf{x})$.

Definition 1.2.20 (Standard boundary domain)**.** A *standard boundary domain with normal* $\mathbf{n} \in \mathbb{S}^{n-1}$ is a bounded domain $D_\mathbf{n} \subset H_\mathbf{n}$, such that the interior $\Gamma_\mathbf{n}$ of $\partial D_\mathbf{n} \cap \partial H_\mathbf{n}$ is nonempty.

Definition 1.2.21 (Quasiconvexity at the boundary)**.** Let $\mathbf{x}_0 \in \partial\Omega_T$, and let $\mathbf{n}$ be the unit normal to $\partial\Omega_T$ at $\mathbf{x}_0$. The stored energy function W is *quasiconvex at the boundary at* $(\mathbf{F}_0,\ \mathbf{n})$ (see Ball and Marsden [BM84]) if for any standard boundary domain $D_\mathbf{n}$, there exists $\mathbf{q} \in \mathbb{R}^n$ such that

$$\int_{D_\mathbf{n}} W(\mathbf{F}_0 + \nabla\boldsymbol{\psi}(\mathbf{x}))\ \mathrm{d}\mathbf{x} - \int_{\Gamma_\mathbf{n}} \mathbf{q}\cdot\boldsymbol{\psi}(\mathbf{x})\ \mathrm{d}S \geq \int_{D_\mathbf{n}} W(\mathbf{F}_0)\ \mathrm{d}\mathbf{x}, \tag{1.37}$$

for every $\boldsymbol{\psi} \in C^1(D_\mathbf{n}, \mathbb{R}^n)$ vanishing in a neighbourhood of $\partial D_\mathbf{n} \setminus \Gamma_\mathbf{n}$.

Remark 1.2.22. Similar to Remark 1.2.18, a scaling argument shows that quasiconvexity at the boundary is independent of the choice of standard boundary domain $D_\mathbf{n}$ (see Mielke and Sprenger [MS98, Remark 2.2]).

Remark 1.2.23. Due to a result by Meyers [Mey65], a necessary condition for $\boldsymbol{\varphi}$ to be a *strong* local minimiser is that Morrey's quasiconvexity condition holds at $\nabla\boldsymbol{\varphi}(\mathbf{x}_0)$ for all (interior) points $\mathbf{x}_0 \in \Omega$. Meyers proves this by taking a strong local minimiser $\boldsymbol{\varphi}$, and considering $\boldsymbol{\varphi} + \mathbf{u}$, where for any given interior point $\mathbf{x}_0 \in \Omega$ and bounded domain D,

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \epsilon\boldsymbol{\psi}\left(\frac{\mathbf{x}-\mathbf{x}_0}{\epsilon}\right) & \text{if } \mathbf{x} \in \mathbf{x}_0 + \epsilon D, \\ 0 & \text{otherwise,}\end{cases}$$

for some $\boldsymbol{\psi} \in C_0^\infty(D, \mathbb{R}^n)$. Since $\boldsymbol{\varphi}$ is a strong local minimiser, $E[\boldsymbol{\varphi} + \mathbf{u}] - E[\boldsymbol{\varphi}] \geq 0$, since $\mathbf{u}$ is uniformly bounded. A change of variables to $\mathbf{y} = \frac{\mathbf{x}-\mathbf{x}_0}{\epsilon}$ followed by taking the limit $\epsilon \to 0$ gives the quasiconvexity condition.

Remark 1.2.24. Ball and Marsden [BM84][11] generalised Meyers' quasiconvexity result to points on the boundary; an additional necessary condition for $\boldsymbol{\varphi}$ to be a strong local minimiser is that for every $\mathbf{x}_0 \in \partial\Omega_T$ with outward normal $\mathbf{n}$, W is quasiconvex at the boundary at $(\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \mathbf{n})$, where the choice of $\mathbf{q}$ in (1.37) must be $\mathbf{q} = \frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))}{\partial \mathbf{F}}\mathbf{n}$. Their method is similar to Meyers' for interior quasiconvexity (see Remark 1.2.23), appropriately modified to account for points on the boundary.

Remark 1.2.25. We may allow our variations $\boldsymbol{\psi}$ in Definition 1.2.17 and Definition 1.2.21 to be in $W^{1,\infty}$ instead of only C^1 provided that W is continuous and finite, and that our set of admissible deformations $\mathcal{A}$ and our set of variations $\mathcal{V}$ allow maps of class $W^{1,\infty}$. This follows from a mollification argument (see [BM84, Remark 1] for further details).

We note the following two results regarding the relationship between W_0, Agmon's condition, and quasiconvexity at the boundary.

Theorem 1.2.26 (Simpson and Spector [SS89, Theorem 1])**.** *Define W_0 by*

$$W_0(\mathbf{F}) := \mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\mathbf{F}, \mathbf{F}], \qquad \text{for all } \mathbf{F} \in M_+^{n\times n}. \tag{1.38}$$

Then for all $\mathbf{x}_0 \in \partial\Omega_T$ with outward unit normal $\mathbf{n}$, W_0 is quasiconvex at the boundary at $(\mathbf{0}, \mathbf{n})$ if and only if

1. *$\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))$ satisfies the Legendre-Hadamard condition;*
2. *the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)), \mathbf{n})$ satisfies Agmon's condition;*
3. *If $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\mathbf{a}\otimes\mathbf{n}, \mathbf{a}\otimes\mathbf{n}] = 0$ for some $\mathbf{a} \in \mathbb{R}^n$ then $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\mathbf{a}\otimes\mathbf{n}] = \mathbf{0}$.*

Theorem 1.2.27 (Mielke and Sprenger [MS98, Main Theorem])**.** *Let $\mathbf{x}_0 \in \partial\Omega_T$ with outward normal $\mathbf{n}$, and let $D_\mathbf{n}$ be a standard boundary domain with normal $\mathbf{n}$. Then*

1. *The following statements are equivalent:*

   (i) W_0 is quasiconvex at the boundary at $(\mathbf{0}, \mathbf{n})$

[11] In fact they define *local* minimisers, show that any strong local minimiser is a local minimiser, and show that any local minimiser is quasiconvex at $\nabla\boldsymbol{\varphi}(\mathbf{x}_0)$ for all $\mathbf{x} \in \Omega$ and quasiconvex at the boundary at $(\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \mathbf{n})$ for all $\mathbf{x}_0 \in \partial\Omega_T$ with outward unit normal $\mathbf{n}$.

(ii) for any $\boldsymbol{\tau} \in \mathbb{R}^n$ perpendicular to $\mathbf{n}$, there exists a Hermitian, positive semidefinite matrix $\mathbf{H}$ such that

$$E_{\boldsymbol{\tau},\mathbf{H}} := \begin{pmatrix} \mathbf{M} & -\mathrm{i}\mathbf{N} + \mathbf{H} \\ \mathrm{i}\mathbf{N}^T + \mathbf{H} & \mathbf{P} \end{pmatrix}$$

is positive semidefinite, where $M_{ij} = C^{ij}_{\alpha\beta} n_\alpha n_\beta$, $N_{ij} = C^{ij}_{\alpha\beta} n_\alpha \tau_\beta$, and $P_{ij} = C^{ij}_{\alpha\beta} \tau_\alpha \tau_\beta$.

2. *If $\mathbf{M}$ is invertible, then either of the above conditions are also equivalent to both of the following statements: for any $\boldsymbol{\tau} \in \mathbb{R}^n$ perpendicular to $\mathbf{n}$, there exists a positive semidefinite matrix $\mathbf{X}$ such that*

   (i) $\mathcal{R}(\mathbf{X}) := (\mathbf{X} + \mathrm{i}\mathbf{N}^T)\mathbf{M}^{-1}(\mathbf{X} - \mathrm{i}\mathbf{N}) - \mathbf{P}$ is negative *semidefinite;*

   (ii) $\mathcal{R}(\mathbf{X}) = \mathbf{0}$.[12]

3. *Let W_0 be given by (1.38). Define, for $\boldsymbol{\psi} \in W^{1,2}(D_\mathbf{n}, \mathbb{R}^n)$ such that $\boldsymbol{\psi} = \mathbf{0}$ on $\partial D_\mathbf{n} \setminus \Gamma_\mathbf{n}$ (in the sense of trace),*

$$Q[\boldsymbol{\psi}] = \int_{D_\mathbf{n}} W_0(\nabla \boldsymbol{\psi}) \, \mathrm{d}\mathbf{x}.$$

   If, in 1., $\mathbf{H}$ can be chosen such that $E_{\boldsymbol{\tau},\mathbf{H}}$ is positive definite, then the quadratic functional Q is coercive.

Remark 1.2.28. The quadratic functional Q is nonnegative for all $\boldsymbol{\psi} \in W^{1,\infty}(D_\mathbf{n}, \mathbb{R}^n)$ vanishing on $\partial D_\mathbf{n} \setminus \Gamma_\mathbf{n}$ (in the sense of trace) if and only if $\mathbf{C}(\nabla \boldsymbol{\varphi}(\mathbf{x}_0))$ is quasiconvex at the boundary at $(\nabla \boldsymbol{\varphi}(\mathbf{x}_0), \mathbf{n})$ (see Mielke and Sprenger [MS98, Theorem 2.5]).

Remark 1.2.29. If a homogeneous deformation induces sufficient compression for Agmon's condition to fail (which implies a loss of weak local minimality), we find smooth solutions to the half-space problem in the form of waves that decay in amplitude in the inward normal direction to the traction surface (see Remark 1.2.15). We associate this with a surface wrinkling instability, but for wrinkling modes to appear in the original nonlinear problem on a bounded domain, a bifurcating branch must exist at this critical deformation. On the other hand, if a homogeneous deformation induces sufficient compression for quasiconvexity at the boundary to fail, this implies a loss of strong local minimality, yet the homogeneous deformation may or may not still be a weak local minimiser. Corresponding variations are uniformly close perturbations which may be singular, and we associate this with a surface creasing instability.

[12] These are known as the Riccati inequality and equality, respectively.

Outline of Chapter 2

In this thesis, we will use the terminology "pure homogeneous" for deformations $\boldsymbol{\varphi}^h : \Omega \to \mathbb{R}^n$ of the form

$$\boldsymbol{\varphi}^h(\mathbf{x}) = \mathbf{D}\mathbf{x}, \tag{1.39}$$

where

$$\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n). \tag{1.40}$$

In Chapter 2, we discuss weak and strong local minimisers (see Definition 1.2.6) in compressible elasticity, focusing on the complementing condition and Agmon's condition (see Definitions 1.2.12 and 1.2.13, and Theorem 1.2.26) corresponding to a pure homogeneous deformation $\boldsymbol{\varphi}^h$. In Section 2.1, in both two and three dimensions, we consider stored energy functions of the general isotropic form

$$W(\mathbf{F}) = \Phi(v_1(\mathbf{F}), \ldots, v_n(\mathbf{F})),$$

under the assumption that Φ is such that $\mathbf{C}(\nabla\boldsymbol{\varphi}^h(\mathbf{x}_0))$ satisfies the strong ellipticity condition (1.30), where $v_i(\mathbf{F})$, for $i = 1, \ldots, n$, are the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$. We also consider a class of examples of Φ of the form

$$\Phi(v_1, \ldots, v_n) = \frac{\mu}{2}\left(v_1^2 + \cdots + v_n^2\right) + H(v_1 \ldots v_n),$$

for some convex function $H \in C^2((0,\infty), \mathbb{R})$, where $\mu > 0$ is a material constant. Our goal will be to obtain a concise algebraic condition, depending only on Φ and the principal stretches of $\boldsymbol{\varphi}^h$, that holds if and only if the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}^h(\mathbf{x}_0)), \mathbf{n})$ satisfies Agmon's condition[13,14] (see Corollary 2.1.10, and Theorem 2.1.15).

In Section 2.2, we study an 'incompressible limit' of each case previously considered, where we take a one-parameter family of isotropic stored energy functions $\Phi(v_1, \ldots, v_n, k)$ depending on an 'incompressibility parameter' k such that $\Phi(v_1, \ldots, v_n, k) \to \infty$ as $k \to \infty$ if $v_1 \ldots v_n \neq 1$. In the general isotropic case in two dimensions, taking this limit will lead us to an isotropic generalization of Biot's critical compression ratio for wrinkling (see (2.50)) for a general stored energy function. For the three dimensional, neo-Hookean example, taking this limit will obtain results agreeing with those for the incompressible neo-Hookean case obtained previously, for example by Chen et al [CYW18] (see (2.56)).

[13] For examples of this, see Propositions 2.1.4 and 2.1.5 for the 2-dimensional case

[14] The same method is followed for an analogous result corresponding to the complementing condition.

1.2.7 Minimisers in incompressible hyperelasticity

In this subsection, we consider an incompressible, isotropic, homogeneous material occupying the region Ω with $\partial\Omega = \partial\Omega_D \cup \partial\Omega_T \cup \partial\Omega_S$, subject to the boundary conditions (1.21) (assuming the boundary data is compatible with the incompressibility constraint (1.2)) and (1.23). We therefore consider minimisers of the functional

$$E^{\mathrm{inc}}[\boldsymbol{\varphi}] = \int_\Omega W^{\mathrm{inc}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}\mathbf{x} - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x})\ \mathrm{d}S(\mathbf{x}) \tag{1.41}$$

under the constraint (1.2), and boundary conditions (1.21) and (1.23), where $\mathbf{t} : \partial\Omega_T \to \mathbb{R}^n$ is a prescribed traction field.

The following definitions and theorems give necessary conditions at the boundary for minimisers in incompressible elasticity, and are due to MacSithigh [Mac05].[15]The following results are stated in n dimensions, where $n = 2$ or 3, although MacSithigh only states and proves these theorems in the case $n = 3$. The simpler case $n = 2$ is not proved in this Thesis, but appears to be a trivial extension of the three-dimensional case.

Define the set of admissible deformations

$$\mathcal{A}^{\mathrm{inc}} := \{\boldsymbol{\varphi} \in C^{1,\rho}(\overline{\Omega}, \mathbb{R}^n) \mid \boldsymbol{\varphi} \text{ satisfies (1.2) and (1.21)}\}. \tag{1.42}$$

Definition 1.2.30 (Incompressible local minimisers)**.** We say the deformation $\boldsymbol{\varphi} \in \mathcal{A}^{\mathrm{inc}}$ is a strong (resp. weak) local minimiser of E^{inc} if there exists $\epsilon > 0$ such that $E^{\mathrm{inc}}(\mathbf{u}) \geq E^{\mathrm{inc}}(\boldsymbol{\varphi})$ for all $\mathbf{u} \in \mathcal{A}^{\mathrm{inc}}$ satisfying $\|\mathbf{u}-\boldsymbol{\varphi}\|_{C^{0,\rho}} < \epsilon$ (resp. $\|\mathbf{u}-\boldsymbol{\varphi}\|_{C^{1,\rho}} < \epsilon$).

Definition 1.2.31 (Incompressible quasiconvexity at the boundary)**.** Let $D_\mathbf{N}$ be a standard boundary domain with normal $\mathbf{N}$ and interior $\Gamma_\mathbf{N}$ of $\partial D_\mathbf{N} \cap \partial H_\mathbf{N}$ (see Definition 1.2.20), and let $\mathbf{F}_0 \in M_1^{n\times n}$. The function W^{inc} is said to be *quasiconvex at the boundary* at $(\mathbf{F}_0, \mathbf{N})$ if there exists a constant vector $\mathbf{t}_0 \in \mathbb{R}^n$ such that

$$\int_{D_\mathbf{N}} W^{\mathrm{inc}}(\nabla\boldsymbol{\xi}(\mathbf{y})\mathbf{F}_0) - W^{\mathrm{inc}}(\mathbf{F}_0)\ \mathrm{d}\mathbf{y} - \int_{\Gamma_\mathbf{N}} \mathbf{t}_0\cdot(\boldsymbol{\xi}(\mathbf{y})-\mathbf{y})\ \mathrm{d}S(\mathbf{y}) \geq 0 \tag{1.43}$$

for all smooth isochoric maps $\boldsymbol{\xi} : D_\mathbf{N} \to \mathbb{R}^n$ such that $\boldsymbol{\xi}(\mathbf{y}) = \mathbf{y}$ near $\partial D_\mathbf{N} \setminus \Gamma_\mathbf{N}$.

Remark 1.2.32. A scaling argument shows that quasiconvexity at the boundary is independent of the choice of standard boundary domain $D_\mathbf{N}$ (see Remark 1.2.22).[16]

[15] MacSithigh uses a domain with the point $x_0 \in \partial\Omega_T$ such that the deformed outward unit normal is $\mathbf{N}(\boldsymbol{\varphi}(\mathbf{x}_0)) = -\mathbf{e}_3$, so discrepancies appearing with a $-$ sign will appear due to this difference.

[16] MacSithigh [Mac05, equation (3.2)] defines quasiconvexity at the boundary using the particular standard boundary domain $D_\mathbf{N} = B_1(0) \cap \{x_n > 0\}$.

Given a deformation $\boldsymbol{\varphi} \in \mathcal{A}^{\text{inc}}$, we assume[17] the domain Ω is such that there exists $\mathbf{x}_0 \in \partial\Omega_T$ such that the outward unit normal to $\boldsymbol{\varphi}(\partial\Omega_T)$ at $\boldsymbol{\varphi}(\mathbf{x}_0)$ is

$$\mathbf{N} := \mathbf{N}(\boldsymbol{\varphi}(\mathbf{x}_0)) = \mathbf{e}_n. \tag{1.44}$$

Define the sets

$$G = \{\mathbf{y} \in \mathbb{R}^n \mid |y_k| < \tfrac{1}{4} \text{ for } k = 1, \ldots, n,\ y_n < 0\}, \tag{1.45}$$

$$\partial G_T = \partial G \cap \{y_n = 0\}, \tag{1.46}$$

$$\partial G_D = \partial G \setminus \partial G_T, \tag{1.47}$$

so that $\partial G = \partial G_D \cup \partial G_T$ is a partition.

Theorem 1.2.33 (MacSithigh [Mac05, Theorem 3.1])**.** *A necessary condition for a deformation $\boldsymbol{\varphi}$ to be a strong local minimiser of E^{inc} is that for any $\mathbf{x}_0 \in \partial\Omega_T$ such that the normal to $\boldsymbol{\varphi}(\partial\Omega_T)$ at $\boldsymbol{\varphi}(\mathbf{x}_0)$ is $\mathbf{N} = \mathbf{e}_n$, W^{inc} is quasiconvex at the boundary at $(\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \mathbf{N})$, with the choice of $\mathbf{t}_0$ in* (1.43) *being $\mathbf{t}_0 = |\nabla\boldsymbol{\varphi}(\mathbf{x}_0)^T\mathbf{N}|\mathbf{t}(\mathbf{x}_0)$, where $\mathbf{t}$ is the traction on $\partial\Omega_T$ in* (1.41)*.*

Remark 1.2.34. In the proof of [Mac05, Theorem 3.1], the comparison of energies is made between an assumed strong local minimiser $\boldsymbol{\varphi}$, and $\mathbf{Z} \circ \boldsymbol{\varphi}$, where for a given point $\mathbf{x}_0 \in \Omega$, and some region D,

$$\mathbf{Z}(\mathbf{y}) = \begin{cases} \epsilon\boldsymbol{\xi}\left(\frac{\mathbf{y}-\boldsymbol{\varphi}(\mathbf{x}_0)}{\epsilon}\right) & \text{if } \mathbf{y} \in (\boldsymbol{\varphi}(\mathbf{x}_0) + \epsilon D) \cap \overline{\boldsymbol{\varphi}(\Omega)} \\ \mathbf{y} & \text{otherwise} \end{cases}$$

where $\boldsymbol{\xi} \in C^\infty(D, \mathbb{R}^3)$ is isochoric and $\boldsymbol{\xi}(\mathbf{y}) = \mathbf{y}$ on ∂D. Then $\mathbf{Z} \circ \boldsymbol{\varphi}$ is isochoric, and uniformly approximates $\boldsymbol{\varphi}$ for small ϵ. Since $\boldsymbol{\varphi}$ is a strong local minimiser, $E^{\text{inc}}[\mathbf{Z} \circ \boldsymbol{\varphi}] - E^{\text{inc}}[\boldsymbol{\varphi}] \geq 0$. Changing to deformed coordinates in the integral for this inequality, and then taking the limit $\epsilon \to 0$, one obtains (1.43), which is the condition of quasiconvexity at the boundary for incompressible elasticity. Note that deriving this condition of quasiconvexity at the boundary involves a composition with an isochoric map, as opposed to the compressible version in Definition 1.2.21, where the variation is additive (see Remark 1.2.23 and Remark 1.2.24).

Remark 1.2.35. Later in this thesis we will relax Definition 1.2.31 to allow $\boldsymbol{\xi}$ to be of class $W^{1,\infty}$.[18]

[17] This is a practice in [Mac05] which we also adopt for convenience.

[18] MacSithigh remarks in [Mac05, pp. 225] that this regularity requirement of $C^{1,\rho}$ maps is more restrictive than necessary.

The incompressible stored energy function $W^{\mathrm{inc}} : M_1^{n\times n} \to \mathbb{R}$ can be extended to a stored energy function $W : M_+^{n\times n} \to \mathbb{R}$, i.e. satisfying

$$W^{\mathrm{inc}}(\mathbf{F}) = W(\mathbf{F}), \quad \text{for all } \mathbf{F} \in M_1^{n\times n}. \tag{1.48}$$

We assume henceforth that $W^{\mathrm{inc}} \in C^2(M_1^{n\times n}, \mathbb{R})$ and that $W \in C^2(M_+^{n\times n}, \mathbb{R})$.

Theorem 1.2.36 (MacSithigh [Mac05, equation (3.15) and preceding arguments])**.** *Let $\boldsymbol{\varphi}$ be a strong local minimiser, and let $\mathbf{x}_0 \in \partial\Omega_T$ be such that the outward unit normal to $\boldsymbol{\varphi}(\partial\Omega_T)$ at $\boldsymbol{\varphi}(\mathbf{x}_0)$ is $\mathbf{N} = \mathbf{e}_n$. Then the functional J, defined by*

$$J[\mathbf{w}] := \int_G \mathbb{K}[\nabla\mathbf{w}, \nabla\mathbf{w}]\ \mathrm{d}\mathbf{y}, \tag{1.49}$$

is nonnegative for all smooth solenoidal $\mathbf{w} : G \to \mathbb{R}^n$ such that $\mathbf{w} = \mathbf{0}$ on ∂G_D, where

$$\begin{aligned} \mathbb{K}[\mathbf{A}_1, \mathbf{A}_2] = \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))}{\partial\mathbf{F}^2}&[\mathbf{A}_1\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \mathbf{A}_2\nabla\boldsymbol{\varphi}(\mathbf{x}_0)] + p\,\mathrm{tr}(\mathbf{A}_1\mathbf{A}_2), \\ &\text{for } \mathbf{A}_1, \mathbf{A}_2 \in \{\mathbf{A} \in M^{n\times n} \mid \mathrm{tr}(\mathbf{A}) = 0\}, \end{aligned} \tag{1.50}$$

and the constant p is such that

$$\left(\frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))}{\partial\mathbf{F}}\nabla\boldsymbol{\varphi}(\mathbf{x}_0)^T - p\mathbb{1}\right)\mathbf{N} = |\nabla\boldsymbol{\varphi}(\mathbf{x}_0)^T\mathbf{N}|\mathbf{t}(\mathbf{x}_0). \tag{1.51}$$

Remark 1.2.37. It is remarked in [Mac05, equation (4.4)] that the functional $J[\mathbf{w}]$ is nonnegative for all smooth solenoidal $\mathbf{w} : G \to \mathbb{R}^n$ such that $\mathbf{w} = \mathbf{0}$ on ∂G_D if and only if the functional

$$K[\mathbf{u}] := \int_G \mathbb{K}[\overline{\nabla\mathbf{u}}, \nabla\mathbf{u}]\ \mathrm{d}\mathbf{y} \tag{1.52}$$

is nonnegative for all (complex) smooth solenoidal $\mathbf{u} : G \to \mathbb{C}^n$ such that $\mathbf{u} = \mathbf{0}$ on ∂G_D.

Define, for $\boldsymbol{\tau} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$M_{ij} = \mathbb{K}^{ij}_{\alpha\beta} n_\alpha n_\beta \quad N_{ij} = i\mathbb{K}^{ij}_{\alpha\beta} n_\alpha \tau_\beta \quad P_{ij} = -\mathbb{K}^{ij}_{\alpha\beta}\tau_\alpha\tau_\beta. \tag{1.53}$$

Theorem 1.2.38 (MacSithigh [Mac05], [Mac07, Theorem 2.1])**.** *The functional K, given by* (1.52)*, is nonnegative for all smooth, solenoidal $\mathbf{u} : G \to \mathbb{C}^n$ such that $\mathbf{u} = \mathbf{0}$ on ∂G_D if and only if the following three conditions are satisfied:*

1. *(Legendre-Hadamard Condition) If $\mathbf{a}\cdot\mathbf{b} = 0$, then*

$$\mathbb{K}[\mathbf{a}\otimes\mathbf{b}, \mathbf{a}\otimes\mathbf{b}] \geq 0; \tag{1.54}$$

2. *(Agmon's Condition) For any $\alpha > 0$, and any nonzero $\boldsymbol{\tau} \in \mathbb{R}^n$ orthogonal to $\mathbf{n}$, the only solution pair $\mathbf{z} : (-\infty, 0] \to \mathbb{C}^n$, $q : (-\infty, 0] \to \mathbb{C}$ such that $\mathbf{z}(t) \to \mathbf{0}$ and $q(t) \to 0$ as $t \to -\infty$ of the system of ordinary differential equations*

$$\mathbf{M}\ddot{\mathbf{z}}(t) + \left(\mathbf{N} + \mathbf{N}^T\right)\dot{\mathbf{z}}(t) + \left(\mathbf{P} - \alpha^2 \mathbb{1}\right)\mathbf{z}(t) + iq(t)\boldsymbol{\tau} + \dot{q}(t)\mathbf{n} = 0, \quad \text{(1.55a)}$$

$$\mathbf{n} \cdot \dot{\mathbf{z}}(t) + i\boldsymbol{\tau} \cdot \mathbf{z}(t) = 0, \quad \text{(1.55b)}$$

$$\mathbf{M}\dot{\mathbf{z}}(0) + \mathbf{N}\mathbf{z}(0) + q(0)\mathbf{n} = 0, \quad \text{(1.55c)}$$

is the zero solution $(\mathbf{z}, q) \equiv (\mathbf{0}, 0)$;

3. *(Supplementary Condition) If $\boldsymbol{\tau} \cdot \mathbf{n} = 0$ and $\mathbb{K}[\boldsymbol{\tau} \otimes \mathbf{n}, \boldsymbol{\tau} \otimes \mathbf{n}] = 0$, then for some scalar function $\pi(\boldsymbol{\tau})$ linear in $\boldsymbol{\tau}$,*

$$\mathbb{K}[\boldsymbol{\tau} \otimes \mathbf{n}] = \pi(\boldsymbol{\tau})\mathbb{1}.$$

Remark 1.2.39. Theorem 1.2.38 is fully stated in [Mac07, Theorem 2.1], but the proof is given in several separate parts throughout [Mac05]. In the latter, it is noted that the system (1.55) is formally the Euler-Lagrange equations for the functional

$$\mathcal{Q}_\alpha^{\boldsymbol{\tau}}[\mathbf{z}] = \int_{-\infty}^0 \dot{\bar{\mathbf{z}}}^T \mathbf{M}\dot{\mathbf{z}} + 2\mathrm{Re}\left(\dot{\bar{\mathbf{z}}}^T \mathbf{N}\mathbf{z}\right) - \bar{\mathbf{z}}^T\left(\mathbf{P} - \alpha^2\mathbb{1}\right)\mathbf{z}\ \mathrm{d}t,$$

under the constraint that $\mathbf{z}$ satisfies (1.55b) see [Mac05, Remark 7.1]. In addition, $\mathcal{Q}_\alpha^{\boldsymbol{\tau}}$ is nonnegative for all $\alpha > 0$ and all $\mathbf{z}$ satisfying (1.55b) if and only if the functional K given by (1.52) is nonnegative for all solenoidal $\mathbf{u}$ with $\mathbf{u} = 0$ on ∂G_D see [Mac05, Proposition 4.2 and equation (4.16)].

Remark 1.2.40. Theorem 1.2.38 serves as an incompressible analogue to Theorem 1.2.26, although the results in [Mac05] show that the nonnegativity of K (or J) follows only if $\boldsymbol{\varphi}$ is a *strong* local minimiser. The fact that this is also a necessary condition for a weak local minimiser is claimed in [Mac05] and [Mac07], but a proof is not given, nor is any indication of how to adapt the proofs of the corresponding results for strong local minimisers to also apply to weak local minimisers.

Outline of Chapter 3

In Chapter 3, we study an incompressible analogue of Chapter 2. In Section 3.1, we show that the nonnegativity of J (or K, see Remark 1.2.37) is a necessary condition for a deformation $\boldsymbol{\varphi}$ to be a *weak* local minimiser of E^{inc} (see Theorem 3.1.5). In Section 3.2, we show that the second variation (defined in Theorem 3.1.3) and J are invariant with respect to the choice of extension W satisfying (1.48). When W^{inc} is isotropic, we may assume that the extension W is also. In this case, there exists a corresponding function Φ satisfying (1.8). In Section 3.3, our main goal is to seek a concise algebraic condition equivalent to (the incompressible version of) Agmon's condition (see Theorem 1.2.38, (2)), which depends only on (the extension) Φ, its derivatives, and the principal stretches $\lambda_1, \ldots, \lambda_n$ of a pure homogeneous deformation $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$ (analogously to Chapter 2). In two and three dimensions, we study Agmon's condition for stored energy functions of the general incompressible isotropic form

$$W^{\text{inc}}(\mathbf{F}) = \Phi^{\text{inc}}(v_1(\mathbf{F}), \ldots, v_n(\mathbf{F})),$$

under the assumption that Φ^{inc} is such that $\mathbb{K}$ (given by (1.50)) satisfies the Legendre-Hadamard condition (1.54), where $v_i(\mathbf{F})$, for $i = 1, \ldots, n$, are the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$. We also consider an example of this where Φ^{inc} is of the form

$$\Phi^{\text{inc}}(v_1, \ldots, v_n) = \frac{\mu}{2}\left(v_1^2 + \cdots + v_n^2\right),$$

where $\mu > 0$ is a material constant. The results we derive here will be shown to agree with those of Chapter 2, Section 2.2 (for the case in two dimensions, see Theorem 3.3.4, and for the case in three dimensions, see Theorem 3.3.11). Furthermore, for the general isotropic case in two dimensions, we show that the result is invariant with respect to the choice of extension Φ (see Remark 3.3.6).

1.2.8 Controllable deformations

We now turn our attention to what are known as controllable deformations.

Definition 1.2.41 (Controllable deformations)**.** A deformation $\boldsymbol{\varphi}$ is said to be controllable if, given any incompressible, isotropic stored energy function W^{inc} and any domain Ω, there exists a scalar function p such that (1.19) holds.

Remark 1.2.42. Some authors define controllable deformations to be deformations that can be supported by surface tractions alone, in either compressible or incompressible elasticity (see, for example, Beatty [Bea84]). In the case of *compressible* elasticity, Ericksen [Eri55] has classified all controllable deformations as homogeneous deformations. Hence, we will only consider the incompressible case for this thesis.

The problem of classifying all controllable deformations possible in any incompressible, isotropic, homogeneous material was proposed, and almost completely solved, by Ericksen in [Eri54]. In addition to homogeneous deformations (called 'family 0'), Ericksen listed four other families of deformations that are controllable (called 'families

1-4'). However, these families form a complete set of solutions except for deformations with all principal invariants (1.4) constant. Many examples satisfying this condition have since been given, with Singh and Pipkin [SP65] finding a 3-parameter family of controllable deformations in $n = 3$ dimensions which seems to generalise all of these known examples, given in terms of deformed cylindrical coordinates (R, Θ, Z), and undeformed cylindrical coordinates (r, θ, z), by

$$R = Ar, \qquad \Theta = B \log r + C\theta, \qquad Z = \frac{z}{A^2 C}. \tag{1.56}$$

Many authors call the class of deformations of the form (1.56) 'family 5'. See Beatty [Bea84] for a detailed history of this problem.

The incompressible double-covering map

A particularly interesting subclass of (1.56) is the case when $B = 0$ and $C = 2$. Writing the deformed cartesian coordinates in terms of undeformed cylindrical polar coordinates, we have

$$\widetilde{\boldsymbol{\varphi}}_{DC\gamma} = \begin{pmatrix} \frac{r}{\sqrt{2\gamma}} \cos(2\theta) \\ \frac{r}{\sqrt{2\gamma}} \sin(2\theta) \\ \gamma z \end{pmatrix}, \tag{1.57}$$

for some scalar $\gamma > 0$. We call a deformation of the form (1.57) an *incompressible double-covering map*, so named since the mapping takes a cylinder with radius 1, height L, and axis of symmetry on the z-axis to a cylinder with radius $\frac{1}{\sqrt{2\gamma}}$, height γL, and the same orientation, covering the image twice. Furthermore, $\widetilde{\boldsymbol{\varphi}}_{DC\gamma}$ maps a half-cylinder to a cylinder, and is one-to-one. To understand the nature of a fully formed crease of an initially flat surface, it is of interest to consider the localised problem at the cusp by studying the double-covering map on a half-cylinder. Indeed, studies such as Silling [Sil91] and Ciarletta [Cia18] suggest that the singularity at a crease can be locally modelled by double-covering map.

The related deformation

$$\widetilde{\boldsymbol{\varphi}} = \begin{pmatrix} \frac{r}{\sqrt{2}} \cos(2\theta) \\ \frac{r}{\sqrt{2}} \sin(2\theta) \end{pmatrix}, \tag{1.58}$$

which can be interpreted as a two-dimensional version of (1.57) with $\gamma = 1$ (where circles are mapped to circles twice over), has been studied by Bevan in [Bev14], where it is shown that the (two-dimensional) double-covering map $\widetilde{\boldsymbol{\varphi}}$ (given by (1.58)) is the global minimiser of the 'neo-Hookean' stored energy $E[\boldsymbol{\varphi}] = \int_B |\nabla \boldsymbol{\varphi}|^2 \, \mathrm{d}\mathbf{x}$ over the unit disk $B \subset \mathbb{R}^2$ for all incompressible maps subject to displacement boundary data $\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}}$ on ∂B. Bevan does not restrict admissible maps to be one-to-one, but the fact that the double-covering map is indeed a global minimiser for *some* variational problem is promising, and motivates further study.

Outline of Chapter 4

In Chapter 4, we consider two problems appropriate for the study of maps that close a half-cylinder

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1,\ x_1 > 0,\ 0 < x_3 < L\} \tag{1.59}$$

to a full cylinder, such as the incompressible double-covering map (1.57). In Section 4.1, we take a general *compressible* isotropic material satisfying the tension-extension inequalities (see (4.12)) occupying the region Ω given by (1.59), with slip boundary condition on its flat top and bottom (half-disk) ends $x_3 = 0, L$, and zero traction on its curved boundary $r = 1$. We impose one of two types of boundary condition on its remaining flat side Γ_2:

Case 1: Dirichlet boundary data $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0$, for some given map $\boldsymbol{\varphi}_0$;

Case 2: 'Self-contact' boundary condition,

$$\begin{gathered}\text{for all } \mathbf{x}, \mathbf{y} \in \Gamma_2 \text{ such that } \boldsymbol{\varphi}^{-1}(\boldsymbol{\varphi}(\mathbf{x})) = \{\mathbf{x}, \mathbf{y}\},\\ \widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\mathbf{n}(\mathbf{x}) + \widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{y}))\mathbf{n}(\mathbf{y}) = 0.\end{gathered}$$

We show that a compressible map of the form

$$\widetilde{\boldsymbol{\varphi}} = \begin{pmatrix} R(r)\cos(\beta\theta + \alpha) \\ R(r)\sin(\beta\theta + \alpha) \\ \gamma z \end{pmatrix}$$

is a 'weak equilibrium solution' (see Definition 4.1.6) if and only if the deformed radius $R(r)$ and the stored energy function Φ satisfy an appropriate boundary value problem (see Theorem 4.1.7). In particular, for the self-contact boundary value problem, the case $\beta = 2$ (a 'compressible double-covering' case) is shown to be a solution to this problem for a wider class of functions Φ than if $\beta < 2$ (see Theorem 4.1.7, "Case 2"). In Section 4.2, we consider *incompressible*, isotropic stored energy functions of the form

$$W^{\text{inc}}(\mathbf{F}) = h^{\text{inc}}(|\mathbf{F}|, |\mathrm{Cof}\mathbf{F}|) \qquad \mathbf{F} \in M_1^{3\times 3},$$

such that h^{inc} is monotone increasing and convex in each argument.[19] We suppose that this material occupies the half-cylinder Ω given by (1.59). Similarly to the previous problem, we impose a slip boundary condition on its flat top and bottom cross section and zero traction on its curved boundary. For the remaining flat boundary, we impose the 'forced self-contact' boundary condition

$$\boldsymbol{\varphi}(0, x_2, x_3) = \boldsymbol{\varphi}(0, -x_2, x_3), \qquad \text{for } 0 < x_2 < 1,\ 0 < x_3 < L.$$

We first consider the two-dimensional analogue, with incompressible deformations on a (unit) half-disk. We show that the two-dimensional double-covering map $\alpha\widetilde{\boldsymbol{\varphi}}$, where $\widetilde{\boldsymbol{\varphi}}$ is given by (1.58), is the unique global minimiser over deformations subject to the constraint

$$\det(\nabla\boldsymbol{\varphi}(\mathbf{x})) = \alpha^2, \qquad \mathbf{x} \in \Omega$$

and satisfying the appropriate boundary conditions for this variational problem (see Theorem 4.2.6). For the three-dimensional case, we extend our result for two dimensions to prove that the double-covering map $\boldsymbol{\varphi}_{DC_\gamma}$, given by (1.57), is the global minimiser for this problem over the set of 'cylindrical' admissible deformations of the form

$$\boldsymbol{\varphi}(\mathbf{x}) = \begin{pmatrix} \varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \\ \gamma x_3 \end{pmatrix}$$

which satisfy the given boundary conditions (see Theorem 4.2.8).

Outline of Chapter 5

In Chapter 5, direct stored energy comparisons are made between pure homogeneous deformations and comparable (in the sense of matching displacement boundary data) crease-like maps. In Section 5.1, we take a rectangular block of compressible 'neo-Hookean' material, and compare the stored energies of a piecewise continuous deformation constructed such that it maps a portion of the free surface to contact itself, and a comparable pure homogeneous deformation. We show that the 'compressible neo-Hookean' stored energy of the crease-like deformation is less than that for the pure homogeneous deformation for large enough strains and suitable dimensions of the block (see Proposition 5.1.6). In Section 5.2, we take a compressible neo-Hookean material occupying the half-space $\{z \in \mathbb{C} \mid \text{Im}(z) < 0\}$, and compare the stored energies of a

[19] Such materials are a particular class of *polyconvex* materials, which were proposed as a suitable model for many materials by Ball [Bal77].

crease-forming deformation constructed using holomorphic maps and a pure homogeneous deformation with matching far-field behaviour. We study the behaviour of this holomorphic map local to the crease, where we show that holomorphic maps do not result in “double-covering behaviour” at the singularity.

Outline of Chapter 6

Finally, in Chapter 6, we discuss some interesting open problems that have arisen during the course of the work presented in Chapters 1-5.

Chapter 2

Necessary conditions for weak local minimisers in compressible, isotropic hyperelasticity

A necessary condition for a deformation $\boldsymbol{\varphi}$ to be a strong local minimiser is that, for any point $\mathbf{x}_0 \in \partial\Omega_T$ with outward unit normal $\mathbf{n}$, W is Quasiconvex at the boundary at $(\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \mathbf{n})$ (see Definition 1.2.21). The following argument of Simpson and Spector ([SS89, Proposition 4.2]) shows how this fact can be used to find further necessary conditions for a deformation $\boldsymbol{\varphi}$ to be a *weak* local minimiser.

It is known that a necessary condition for a deformation $\boldsymbol{\varphi}$ to be a weak local minimiser is that $\delta^2 E(\boldsymbol{\varphi})[\mathbf{v}] \geq 0 = \delta^2 E(\boldsymbol{\varphi})[\mathbf{0}]$ for all $\mathbf{v} \in \mathcal{V}$, where $\delta^2 E(\boldsymbol{\varphi})[\cdot]$ is the second variation given by (1.31), and $\mathcal{V}$ is given by (1.27). This implies that $\mathbf{v} \equiv \mathbf{0}$ is a global minimiser of $\delta^2 E(\boldsymbol{\varphi})[\cdot]$. A necessary condition for this is that, for any $\mathbf{x}_0 \in \partial\Omega_T$ with outward normal $\mathbf{n}$, W_0 is quasiconvex at the boundary at $(\mathbf{0}, \mathbf{n})$ (see Definition 1.2.21), where W_0 is given by (1.38). By Theorem 1.2.26, this holds if and only if the following three conditions are satisfied:

1. $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))$ satisfies the Legendre-Hadamard condition;

2. the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)), \mathbf{n})$ satisfies Agmon's condition;

3. If $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\mathbf{a} \otimes \mathbf{n}, \mathbf{a} \otimes \mathbf{n}] = 0$ for some $\mathbf{a} \in \mathbb{R}^n$ then $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\mathbf{a} \otimes \mathbf{n}] = \mathbf{0}$.

We will assume that the elasticity tensor $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))$ (given by (1.28)) satisfies the strong ellipticity condition (1.30), from which it follows that parts (1) and (3) of Theorem 1.2.26 are satisfied (see Remark 2.1.1). Overall, we have that if $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))$ satisfies the strong ellipticity condition (1.30), then for any $\mathbf{x}_0 \in \partial\Omega_T$ with outward unit normal $\mathbf{n}$, W_0 is quasiconvex at the boundary at $(\mathbf{0}, \mathbf{n})$ (which is a necessary condition for $\boldsymbol{\varphi}$ to be a weak local minimiser) if and only if the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)), \mathbf{n})$ satisfies Agmon's condition (see Definition 1.2.13).

It is noted by many authors (e.g. see [SS87], [NMMP11]) that the complementing condition and Agmon's Condition, which are stated as boundary value problems, are actually algebraic conditions on the elasticity tensor $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))$. We will study the complementing condition and Agmon's condition in two and three dimensions for isotropic materials under a pure homogeneous deformation $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$, given by

$$\boldsymbol{\varphi}^h(\mathbf{x}) = \mathbf{D}\mathbf{x}, \tag{2.1}$$

where

$$\mathbf{D} = \mathrm{diag}(\lambda_1, \dots, \lambda_n). \tag{2.2}$$

Our goal in each case will be to replace the complementing condition or Agmon's condition with a concise, purely algebraic condition depending only on $\mathbf{C}(\nabla\boldsymbol{\varphi}^h(\mathbf{x}_0))$ and the principal stretches of $\boldsymbol{\varphi}^h(\mathbf{x}_0)$.

We remind the reader that the Einstein summation convention is *not* assumed for this chapter.

2.1 Agmon's condition for compressible, isotropic hyperelasticity

Consider a hyperelastic, isotropic, homogeneous material in n dimensions, where $n = 2$ or 3, occupying a block region $\Omega = (0, L_1) \times \cdots \times (0, L_n) \subset \mathbb{R}^n$ in its reference state. Let $\Phi \in C^2((0,\infty)^n, \mathbb{R})$ be its isotropic stored energy function. Suppose, for some $\lambda_1, \dots, \lambda_{n-1}$, that this material is subjected to a slip boundary condition on $\partial\Omega_S \subset \partial\Omega$, given by

if $\mathbf{n = 2}$, $\partial\Omega_S = \{0, L_1\} \times (0, L_2)$, with slip boundary condition

$$\varphi_1(0, x_2) = 0, \quad \varphi_1(L_1, x_2) = \lambda_1 L_1 \quad \text{for } x_2 \in (0, L_2), \tag{2.3}$$

if $\mathbf{n = 3}$, $\partial\Omega_S = (\{0, L_1\} \times (0, L_2) \times (0, L_3)) \cup ((0, L_1) \times \{0, L_2\} \times (0, L_3))$, with slip boundary condition

$$\begin{cases} \varphi_1(0, x_2, x_3) = 0, \quad \varphi_1(L_1, x_2, x_3) = \lambda_1 L_1 & \text{for } (x_2, x_3) \in (0, L_2) \times (0, L_3) \\ \varphi_2(x_1, 0, x_3) = 0, \quad \varphi_1(x_1, L_2, x_3) = \lambda_2 L_2 & \text{for } (x_1, x_3) \in (0, L_1) \times (0, L_3) \end{cases} \tag{2.4}$$

and the remaining part of the boundary $\partial\Omega_T = \partial\Omega \setminus \partial\Omega_S$ is left free (see Figure 2-1)[1]. We let λ_n be such that the pure homogeneous deformation $\boldsymbol{\varphi}^h$, given by (2.1), satisfies the natural 'stress-free' boundary condition (1.22) with traction $\mathbf{t} \equiv \mathbf{0}$, so that $\boldsymbol{\varphi}^h$ is a solution of the mixed boundary value problem (1.16) and (1.21)-(1.24) (with $\partial\Omega_D = \varnothing$). Let $\mathbf{x}_0 \in \partial\Omega_T$ be a point on the upper surface $x_n = L_n$, so that the normal to $\partial\Omega_T$ at $\mathbf{x}_0$ is $\mathbf{n} = \mathbf{e}_n$.[2]

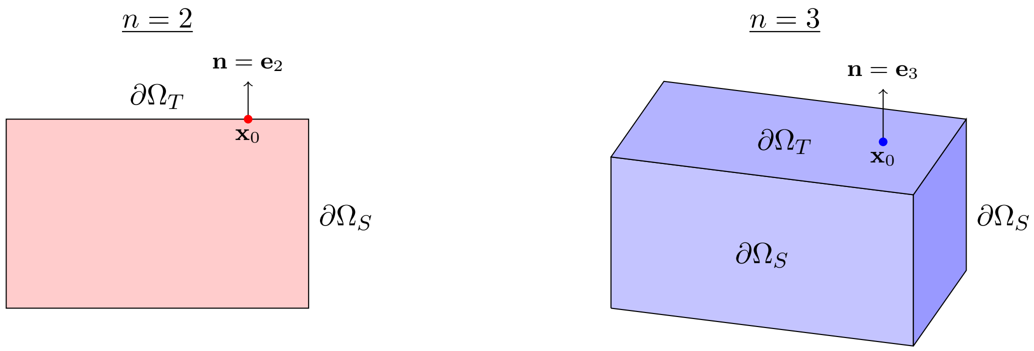


Figure 2-1: The region Ω in two and three dimensions.

We will use the notation[3] (with no sum on repeated indices)

$$\begin{aligned} \Phi_i &= \Phi_{,i}(\lambda_1,\ldots,\lambda_n), & \Phi_{ij} &= \Phi_{,ij}(\lambda_1,\ldots,\lambda_n), & i,j &= 1,\ldots,n, \\ \Psi_{ij} &= \frac{\lambda_i\Phi_i - \lambda_j\Phi_j}{\lambda_i^2-\lambda_j^2}, & \Theta_{ij} &= \frac{\lambda_j\Phi_i - \lambda_i\Phi_j}{\lambda_i^2-\lambda_j^2}, & i &\neq j, \end{aligned} \tag{2.5}$$

and (see Definition 1.28)

$$\mathbf{C} := \mathbf{C}(\nabla\boldsymbol{\varphi}^h(\mathbf{x}_0)) = \frac{\partial^2 W(\mathbf{D})}{\partial \mathbf{F}^2} = \left(\frac{\partial^2 W(\mathbf{D})}{\partial F_{i\alpha}\partial F_{j\beta}}\right), \tag{2.6}$$

where $\mathbf{D}$ is given by (2.2). Furthermore, we assume that Φ is such that for any $\lambda_1,\ldots,\lambda_n$, the tensor $\mathbf{C}$ is strongly elliptic: that is, $\mathbf{C}(\mathbf{F})$ is strongly elliptic at $\mathbf{F} = \nabla\boldsymbol{\varphi}^h(\mathbf{x}_0) = \mathbf{D}$. Note that $\mathbf{C}$ is independent of the point $\mathbf{x}_0 \in \partial\Omega_T$, so the inequality (1.30) holds at one point $\mathbf{x}_0 \in \partial\Omega_T$ if and only if it holds for all $\mathbf{x}_0 \in \partial\Omega_T$.

Remark 2.1.1. Note that since the elasticity tensor $\mathbf{C}$ is strongly elliptic, parts (1) and (3) of Theorem 1.2.26 are satisfied.

We will make frequent use of the following result from Ball.

[1] That is, $\partial\Omega_D = \varnothing$ (see Remark 1.2.5)

[2] All results in this chapter would also follow if we were to choose $\mathbf{x}_0$ on the lower surface $x_n = 0$, so that the normal to $\partial\Omega_T$ at $\mathbf{x}_0$ would be $\mathbf{n} = -\mathbf{e}_n$.

[3] In the case when $\lambda_i = \lambda_j$ for some $i \neq j$, one interprets Ψ_{ij} and Θ_{ij} as a limit $\lambda_j \to \lambda_i$. See also [SS08a, equation (3.4) and specifically footnote 11].

Theorem 2.1.2 (Ball [Bal84, Theorem 6.4]). *Let W be isotropic, and let Φ satisfy* (1.8). *Let* $\mathbf{D}$ *be given by* (2.2)*, with all $\lambda_i > 0$ for $i = 1, \ldots, n$. Then if* $\Phi \in C^1((0,\infty)^n, \mathbb{R})$,

$$\widehat{\mathbf{S}}(\mathbf{D}) \cdot \mathbf{G} = \sum_{i=1}^{n} \Phi_i G_{ii}, \quad \text{for all } \mathbf{G} \in M^{n\times n}, \tag{2.7}$$

where $\widehat{\mathbf{S}}$ is given by (1.12)*, and if* $\Phi \in \mathbf{C}^2((0,\infty)^n, \mathbb{R})$,

$$\mathbf{C}[\mathbf{G},\mathbf{G}] = \sum_{i,j=1}^{n} \Phi_{ij} G_{ii} G_{jj} + \sum_{i\neq j} \left(\Psi_{ij} G_{ij}^2 + \Theta_{ij} G_{ij} G_{ji} \right), \quad \text{for all } \mathbf{G} \in M^{n\times n}, \tag{2.8}$$

where $\mathbf{C}$ *is given by* (2.6)*, and Φ_i, Φ_{ij}, Ψ_{ij}, and Θ_{ij} are given by* (2.5).

Note that, by (2.7), (1.24) holds trivially, and (1.22) holds if and only if

$$\Phi_n = 0. \tag{2.9}$$

Therefore, for $n = 2$ or 3, given $\lambda_1, \ldots, \lambda_n$ which satisfy the boundary conditions (2.3) or (2.4) (respectively), and (2.9), it is of interest to check whether the pair $(\mathbf{C}, \mathbf{n})$ satisfies the complementing condition or Agmon's condition.

Combining Remark 2.1.1 and Theorem 1.2.26, we have that W_0 is quasiconvex at the boundary (which is a necessary condition for $\boldsymbol{\varphi}^h$ to be a weak local minimiser) if and only if the pair $(\mathbf{C}, \mathbf{n})$ satisfies Agmon's condition. The focus of this section will be to find necessary and sufficient conditions for the pair $(\mathbf{C}, \mathbf{n})$ to satisfy Agmon's condition, and therefore for quasiconvexity at the boundary of W_0. Since the complementing condition is almost identical as a boundary value problem, and closely related, it is natural to include it in the present study.

2.1.1 The two dimensional case

General isotropic stored energy functions

The following results are from Knowles and Sternberg [KS76], Davies [Dav89], and Mielke and Sprenger [MS98], respectively.

Proposition 2.1.3 (Knowles and Sternberg [KS76, equation (2.42)]). *Let W be isotropic, and let $\Phi \in C^2((0,\infty)^2, \mathbb{R})$ satisfy* (1.8). *Let* $\mathbf{C}$ *be given by* (2.6). *Then* $\mathbf{C}$ *is strongly elliptic if and only if*

$$\Phi_{11} > 0, \quad \Phi_{22} > 0, \quad \Psi_{12} > 0, \quad \Psi_{12} + \sqrt{\Phi_{11}\Phi_{22}} > |\Phi_{12} + \Theta_{12}|. \tag{2.10}$$

Proposition 2.1.4 (Davies [Dav89, Theorem 8.1]). *Let W be isotropic, and let Φ satisfy* (1.8)*. Let λ_1, λ_2 satisfy* (2.9)*, and let $\mathbf{C}$, given by* (2.6)*, be strongly elliptic. Then the pair* $(\mathbf{C}, \mathbf{n})$ *satisfies the complementing condition if and only if*

$$\Psi_{12}(\Phi_{11}\Phi_{22} - \Phi_{12}^2) + \sqrt{\Phi_{11}\Phi_{22}}(\Psi_{12}^2 - \Theta_{12}^2) \neq 0, \tag{2.11}$$

Proposition 2.1.5 (Mielke and Sprenger [MS98, Example 5.3]). *Let W be isotropic, and let Φ satisfy* (1.8)*. Let λ_1, λ_2 satisfy* (2.9)*, and let $\mathbf{C}$, given by* (2.6)*, be strongly elliptic. Then the pair* $(\mathbf{C}, \mathbf{n})$ *satisfies Agmon's condition if and only if*

$$\Psi_{12}(\Phi_{11}\Phi_{22} - \Phi_{12}^2) + \sqrt{\Phi_{11}\Phi_{22}}(\Psi_{12}^2 - \Theta_{12}^2) \geq 0. \tag{2.12}$$

Remark 2.1.6. Knowles and Sternburg specifically state (2.10) only in the case $\lambda_1 \neq \lambda_2$ [KS76, equation (2.42)]. In the case $\lambda_1 = \lambda_2$, Knowles and Sternburg prove that strong ellipticity holds if and only if the first three inequalities of (2.10) hold. We have stated all four inequalities of (2.10) in both cases, since the last inequality follows from the previous three in the case $\lambda_1 = \lambda_2$ (see also [SS08a, Proposition 3.2 (i)]).

Remark 2.1.7. Thompson [Tho69, equation (31)] also gives (2.11) in a form which was without the use of (2.7) and (2.8). Propositions 2.1.4 and 2.1.5 were also proved by Simpson and Spector [SS08a, Proposition 3.2 (ii) and (iii), respectively].

Equations (2.11) and (2.12) are both examples of simple algebraic conditions depending only on Φ and the stretches λ_1, λ_2 for which we can verify whether the pair $(\mathbf{C}, \mathbf{n})$ satisfies the complementing condition or Agmon's condition, respectively, given the assumption of strong ellipticity of $\mathbf{C}$.

Compressible neo-Hookean stored energy functions

Consider the example isotropic stored energy function Φ of the form

$$\Phi(\lambda_1, \lambda_2) = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) + H(\lambda_1\lambda_2), \tag{2.13}$$

where $H \in C^2((0, \infty), \mathbb{R})$ is some convex function (this is known as a "compressible neo-Hookean" material,[4] see Ciarlet [Cia88, Section 4.10] for the general class as stated above, and Blatz [Bla71, equation (48)] and Simpson and Spector [SS08a, §8] for examples of specific forms for H).

[4] Also sometimes referred to as an "isotropic Hadamard material", see for example Ball and Marsden [BM84, equation (3.14)].

Remark 2.1.8. Many applications make use of the seemingly more general function

$$\Phi(\lambda_1, \lambda_2) = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2) + H(\lambda_1\lambda_2),$$

where $\mu > 0$ is a material constant (the shear modulus). However, one can rewrite $H = \mu\widetilde{H}$, which results in the constant μ trivially factoring out of the following calculations. Hence, we will only consider the simpler case $\mu = 1$.

Remark 2.1.9. A simple calculation shows that the inequalities (2.10) hold when Φ is given by (2.13) for any λ_1, λ_2, and any convex, twice-differentiable function H. Therefore, $\mathbf{C}$ is strongly elliptic in this case.

Since we are assuming λ_1 and λ_2 are such that $\boldsymbol{\varphi}^h$ is an equilibrium solution, (2.9) simplifies to

$$H'(\lambda_1\lambda_2) = -\frac{\lambda_2}{\lambda_1}. \tag{2.14}$$

This gives rise to the following corollary of Proposition 2.1.5.

Corollary 2.1.10. *Suppose Φ is of the form* (2.13)*, and that λ_1, λ_2 satisfy* (2.14)*. Then the pair* $(\mathbf{C}, \mathbf{n})$ *satisfies Agmon's Condition if and only if*

$$(\lambda_1^2 + 3\lambda_2^2)s + \left(1 - \left(\frac{\lambda_2}{\lambda_1}\right)^2\right)\left(1 + \sqrt{(1 + \lambda_1^2 s)(1 + \lambda_2^2 s)}\right) \geq 0, \tag{2.15}$$

where

$$s = H''(\lambda_1\lambda_2). \tag{2.16}$$

Here we have kept H general, but specific choices of H will simplify (2.15) further; more details on the properties of H and how they affect the results will be discussed in a later section. Note that (2.15) only depends on the stretches λ_1, λ_2, and the second derivative of H (evaluated at $\lambda_1\lambda_2$).

2.1.2 The three dimensional case

Many studies already exist of the complementing condition and Agmon's condition in three dimensions for a pure homogeneous deformation of the form (2.1) (see, for example, [NMMP11], [NMMP12], for specifically three dimensions, and [SS87], [SS89], and [MS98] for the more general case in $n \in \mathbb{N}$ dimensions). However, none of these studies include the case which *allow both stretches λ_1 and λ_2 to be independent* (with λ_3 fixed by (2.9)), with the aim to find a concise algebraic condition equivalent to the complementing or Agmon's condition[5] (i.e. a three-dimensional analogue to Propositions

[5]There exist examples of equibiaxial strain in three dimensions, see for example [NMMP12], §4 and §5. This is the specific case $\lambda_1 = \lambda_2$.

2.1.4 and 2.1.5). To our knowledge, the only known study of compressible half-space instability in three dimensions with arbitrary λ_1 and λ_2 is that of Usmani and Beatty [UB74], where conditions for the existence of wave-like perturbations on the half-space are found for a particular compressible neo-Hookean material. The condition they obtain is an example of a concise algebraic simplification of the complementing condition in three dimensions for a compressible neo-Hookean material (see Remark 2.1.16). To our knowledge, the half-space problem associated to Agmon's condition has not previously been studied for a compressible neo-Hookean material in three dimensions.[6] This problem will be studied later in this subsection.

In this subsection, we will study Agmon's condition in three dimensions. The general isotropic case has proved to be too algebraically cumbersome to obtain a concise result. However, we will take a compressible neo-Hookean example (see the two dimensional example where Φ is given by (2.13)) which is simple enough to obtain a result analogous to Corollary 2.1.10.

General isotropic stored energy functions

Define for $\boldsymbol{\tau} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$,

$$\begin{aligned} M_{ij} &= C^{ij}_{33}, \\ N_{ij} &= \mathrm{i}(\tau_1 C^{ij}_{31} + \tau_2 C^{ij}_{32}), \\ P_{ij} &= -[\tau_1^2 C^{ij}_{11} + \tau_1\tau_2(C^{ij}_{12} + C^{ij}_{21}) + \tau_2^2 C^{ij}_{22}]. \end{aligned}$$

By isotropy, and (2.8), $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{P}$ take the form

$$\mathbf{M} = \begin{pmatrix} \Psi_{13} & 0 & 0 \\ 0 & \Psi_{23} & 0 \\ 0 & 0 & \Phi_{33} \end{pmatrix}, \tag{2.17}$$

$$\mathbf{N} = \mathrm{i} \begin{pmatrix} 0 & 0 & \Theta_{13}\tau_1 \\ 0 & 0 & \Theta_{23}\tau_2 \\ \Phi_{13}\tau_1 & \Phi_{23}\tau_2 & 0 \end{pmatrix}, \tag{2.18}$$

$$\mathbf{P} = -\begin{pmatrix} \Phi_{11}\tau_1^2 + \Psi_{12}\tau_2^2 & (\Phi_{12} + \Theta_{12})\tau_1\tau_2 & 0 \\ (\Phi_{12} + \Theta_{12})\tau_1\tau_2 & \Psi_{12}\tau_1^2 + \Phi_{22}\tau_2^2 & 0 \\ 0 & 0 & \Psi_{13}\tau_1^2 + \Psi_{23}\tau_2^2 \end{pmatrix}. \tag{2.19}$$

Lemma 2.1.11. *Let* $\mathbf{n} = \mathbf{e}_3$*, and suppose* $\mathbf{C}$ *is strongly elliptic. Then for any* $\alpha \geq 0$*,*

[6] Usmani and Beatty's study in [UB74] is closely related, but is technically a verification of the complementing condition.

there exist nontrivial solutions to (1.35a) *of the form* (1.36) *which decay to zero as* $x_3 \to -\infty$ *only if there exist 3 roots* $m = m_i(\boldsymbol{\tau}, \alpha)$, $i = 1, 2, 3$, *with* $\mathrm{Re}(m_i(\boldsymbol{\tau}, \alpha)) > 0$ *of the equation*

$$\det \boldsymbol{\chi}(m) = 0, \tag{2.20}$$

where

$$\boldsymbol{\chi}(m) = m^2\mathbf{M} + m(\mathbf{N} + \mathbf{N}^T) + \mathbf{P} - \alpha^2 \mathbb{1}, \tag{2.21}$$

and $\mathbf{M}$, $\mathbf{N}$, *and* $\mathbf{P}$ *are given by* (2.17), (2.18), *and* (2.19), *respectively.*

Proof. We seek solutions to

$$\mathrm{div}\,(\mathbf{C}[\nabla \mathbf{v}]) = \alpha^2 \mathbf{v} \qquad \text{for } x_3 < 0 \tag{2.22}$$

of the form

$$\mathbf{v}(\mathbf{x}) = \mathrm{Re}(\mathbf{z}(x_3) e^{\mathrm{i}(\tau_1 x_1 + \tau_2 x_2)}) \tag{2.23}$$

for some function[7] $\mathbf{z} : (-\infty, 0] \to \mathbb{C}$. When substituting $\mathbf{z}(x_3)e^{\mathrm{i}(\tau_1 x_1 + \tau_2 x_2)}$ into (2.22), we obtain

$$\mathbf{M}\frac{\mathrm{d}^2\mathbf{z}}{\mathrm{d}x_3^2} + (\mathbf{N} + \mathbf{N}^T)\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x_3} + \left(\mathbf{P} - \alpha^2 \mathbb{1}\right)\mathbf{z} = 0 \qquad \text{for } x_3 < 0, \tag{2.24a}$$

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x_3} + \mathbf{N}\mathbf{z} = 0 \qquad \text{on } x_3 = 0, \tag{2.24b}$$

Solutions to this system of 2^{nd} order ordinary differential equations are of the form $\mathbf{z}(x_3) = \mathbf{A}e^{mx_3}$, $m \in \mathbb{C}$, $\mathbf{A} \in \mathbb{C}^3$. Substituting this into (2.24a) gives

$$(m^2\mathbf{M} + m(\mathbf{N} + \mathbf{N}^T) + \mathbf{P} - \alpha^2 \mathbb{1})\mathbf{A} = 0,$$

which has nontrivial solutions $\mathbf{A}$ only if the polynomial equation (2.20) is satisfied. Since $\mathbf{C}$ is strongly elliptic, a result from Simpson [Sim19, Theorem 2.4 (ii) and (iii)] implies that (2.20) has three roots with positive real part, and three roots with negative real part. For solutions to decay to zero as $x_3 \to -\infty$, we only consider the three roots $m_1(\boldsymbol{\tau}, \alpha)$, $m_2(\boldsymbol{\tau}, \alpha)$, and $m_3(\boldsymbol{\tau}, \alpha)$, with $\mathrm{Re}(m_i(\boldsymbol{\tau}, \alpha)) > 0$. □

To verify Agmon's condition, we not only need to find the roots for the general solution to (2.24a), but also to find a nontrivial solution satisfying (2.24b). In doing so, we should obtain a purely algebraic condition on the elasticity tensor $\mathbf{C}$, similar to that of (2.12). The method would likely be similar to that of Simpson and Spector's two-

[7] We have omitted the negative sign in the argument of z appearing in (1.36), and have accordingly changed the domain to make calculating derivatives in x_3 easier.

dimensional example [SS08a]. Unfortunately, this approach has proven to be technically too complex.

Mielke and Sprenger [MS98] give an alternate condition equivalent to quasiconvexity at the boundary of $\mathbf{C}$, which in this case simplifies to solving a 3×3 Riccati equation (see Theorem 1.2.27). Although this method is an alternative technique to the above, it also seems too difficult to simplify to a concise algebraic condition in general.

Compressible neo-Hookean stored energy functions

Further progress can be made with the compressible three-dimensional case by considering materials with an isotropic stored energy function given by

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) + H(\lambda_1\lambda_2\lambda_3), \tag{2.25}$$

where $H : (0, \infty) \to (0, \infty)$ is some convex, twice-differentiable function. Remark 2.1.8 also applies to this three-dimensional case, hence the lack of a material constant in (2.25).

Remark 2.1.12. Similar to the case in two dimensions when Φ is given by (2.13), one can easily show that the inequalities (2.10) hold when Φ is given by (2.25) for any $\lambda_1, \lambda_2, \lambda_3$, and any convex, twice-differentiable function H. Hence, $\mathbf{C}$ is strongly elliptic.

Note that, in the case $n = 3$ with Φ given by (2.25), we have that (2.9) reduces to the requirement that λ_1, λ_2, and λ_3 satisfy

$$H'(\lambda_1\lambda_2\lambda_3) = -\frac{\lambda_3}{\lambda_1\lambda_2}. \tag{2.26}$$

We have the following lemma, which is an example of Lemma 2.1.11 in the case where Φ is of the form (2.25).

Lemma 2.1.13. *Let $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$ be a pure homogeneous deformation given by (2.1). Suppose the isotropic stored energy function Φ is of the form (2.25). Let $\alpha \geq 0$, $\boldsymbol{\tau} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and let $\boldsymbol{\chi}(m)$ be given by (2.21). Then the roots of $\det(\boldsymbol{\chi}(m)) = 0$ with positive real part are given by the repeated root*

$$m_1(\boldsymbol{\tau}, \alpha) = \sqrt{\alpha^2 + |\boldsymbol{\tau}|^2}, \tag{2.27}$$

and the root

$$m_3(\boldsymbol{\tau}, \alpha) = \sqrt{\frac{\alpha^2 + |\boldsymbol{\tau}|^2 + \lambda_3^2 s(\lambda_2^2\tau_1^2 + \lambda_1^2\tau_2^2)}{1 + \lambda_1^2\lambda_2^2 s}}, \tag{2.28}$$

where
$$s = H''(\lambda_1\lambda_2\lambda_3) > 0. \tag{2.29}$$

Proof. Since Φ is of the form (2.25), by (2.17), (2.18), and (2.19), we have that $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{P}$ take the simplified form

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+(\lambda_1\lambda_2)^2 s \end{pmatrix}, \tag{2.30}$$

$$\mathbf{N} = \mathrm{i} \begin{pmatrix} 0 & 0 & \frac{\lambda_3}{\lambda_1\lambda_2}\tau_1\lambda_2 \\ 0 & 0 & \frac{\lambda_3}{\lambda_1\lambda_2}\tau_2\lambda_1 \\ -\frac{\lambda_3}{\lambda_1\lambda_2}\tau_1\lambda_2 + \frac{(\lambda_1\lambda_2\lambda_3)^2\tau_1}{\lambda_1\lambda_3}s & -\frac{\lambda_3}{\lambda_1\lambda_2}\tau_2\lambda_1 + \frac{(\lambda_1\lambda_2\lambda_3)^2\tau_2}{\lambda_2\lambda_3}s & 0 \end{pmatrix}, \tag{2.31}$$

$$\mathbf{P} = -\begin{pmatrix} \tau_1^2+\tau_2^2+(\lambda_1\lambda_2\lambda_3)^2\frac{\tau_1^2}{\lambda_1^2}s & (\lambda_1\lambda_2\lambda_3)^2\frac{\tau_1\tau_2}{\lambda_1\lambda_2}s & 0 \\ (\lambda_1\lambda_2\lambda_3)^2\frac{\tau_1\tau_2}{\lambda_1\lambda_2}s & \tau_1^2+\tau_2^2+(\lambda_1\lambda_2\lambda_3)^2\frac{\tau_2^2}{\lambda_2^2}s & 0 \\ 0 & 0 & \tau_1^2+\tau_2^2 \end{pmatrix}, \tag{2.32}$$

where $s = H''(\lambda_1\lambda_2\lambda_3) > 0$. Hence for $\alpha \geq 0$ and $\boldsymbol{\tau} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, by (2.21),

$$\boldsymbol{\chi}(m) = (m^2 - \tau_1^2 - \tau_2^2 - \alpha^2)\mathbb{1} + (\lambda_1\lambda_2\lambda_3)^2 s \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ \frac{m}{\lambda_3} \end{pmatrix} \otimes \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ \frac{m}{\lambda_3} \end{pmatrix}. \tag{2.33}$$

To solve $\det\boldsymbol{\chi}(m) = 0$, we have

$$\begin{aligned} 0 &= \det\left[(m^2 - \tau_1^2 - \tau_2^2 - \alpha^2)\mathbb{1} + (\lambda_1\lambda_2\lambda_3)^2 s \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ \frac{m}{\lambda_3} \end{pmatrix} \otimes \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ \frac{m}{\lambda_3} \end{pmatrix}\right] \\ &= (m^2 - \tau_1^2 - \tau_2^2 - \alpha^2)^2\left[(m^2 - \tau_1^2 - \tau_2^2 - \alpha^2) + (\lambda_1\lambda_2\lambda_3)^2 s\left(\frac{m^2}{\lambda_3^2} - \frac{\tau_1^2}{\lambda_1^2} - \frac{\tau_2^2}{\lambda_2^2}\right)\right]. \end{aligned}$$

So our three roots with positive real part are the doubly repeated root $m_1(\boldsymbol{\tau}, \alpha)$ given by (2.27), and $m_3(\boldsymbol{\tau}, \alpha)$, given by (2.28). □

Remark 2.1.14. For given $\alpha \geq 0$ and $\boldsymbol{\tau} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, if $\boldsymbol{\tau}$ is in fact a unit vector, we have that m_1 and m_3 simplify to
$$\widetilde{m}_1(\alpha) = \sqrt{\alpha^2 + 1}, \tag{2.34}$$

and
$$\widetilde{m}_3(\boldsymbol{\tau}, \alpha) = \sqrt{\frac{\alpha^2 + 1 + \lambda_3^2 s(\lambda_2^2\tau_1^2 + \lambda_1^2\tau_2^2)}{1 + \lambda_1^2\lambda_2^2 s}}, \tag{2.35}$$

respectively.

Theorem 2.1.15. *Let* Φ *be of the form* (2.25) *(so that the elasticity tensor* $\mathbf{C}$ *is strongly elliptic), and let* $\lambda_1, \lambda_2, \lambda_3$ *satisfy* (2.26)*. Let* $\widetilde{m}_1$ *be given by* (2.34)*, and let* $\widetilde{m}_3$ *be given by* (2.35)*. Then*

1. *(Usmani and Beatty [UB74, equation (3.16)]) the pair* $(\mathbf{C}, \mathbf{n})$ *satisfies the complementing condition if and only if* $\lambda_1 = \lambda_2 = \lambda_3$*, or for all* $\boldsymbol{\tau} \in \mathbb{S}^1$ *such that* $\widetilde{m}_3(\boldsymbol{\tau}, 0) \neq 1$*,*
$$\left(\frac{\tau_1^2}{\lambda_1^2} + \frac{\tau_2^2}{\lambda_2^2} + \frac{1}{\lambda_3^2}\right)^2 - \frac{4\widetilde{m}_3(\boldsymbol{\tau}, 0)}{\lambda_3^2}\left(\frac{\tau_1^2}{\lambda_1^2} + \frac{\tau_2^2}{\lambda_2^2}\right) \neq 0; \tag{2.36}$$

2. *the pair* $(\mathbf{C}, \mathbf{n})$ *satisfies Agmon's condition if and only if for all* $\alpha > 0$*, and all* $\boldsymbol{\tau} \in \mathbb{S}^1$ *such that* $\widetilde{m}_1(\alpha) \neq \widetilde{m}_3(\boldsymbol{\tau}, \alpha)$*,*
$$\left(\frac{\tau_1^2}{\lambda_1^2} + \frac{\tau_2^2}{\lambda_2^2} + \frac{\widetilde{m}_1(\alpha)^2}{\lambda_3^2}\right)^2 - \frac{4\widetilde{m}_1(\alpha)\widetilde{m}_3(\boldsymbol{\tau}, \alpha)}{\lambda_3^2}\left(\frac{\tau_1^2}{\lambda_1^2} + \frac{\tau_2^2}{\lambda_2^2}\right) \neq 0. \tag{2.37}$$

Proof. To check either the complementing condition or Agmon's condition, we suppose $\alpha \geq 0$ and $\boldsymbol{\tau} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and we seek solutions to (2.24a). Since Φ is of the form (2.25), we have that $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{P}$ are given by (2.30), (2.31), and (2.32), respectively. We have two cases to consider:

Nondegenerate case Suppose α and $\boldsymbol{\tau}$ are such that $m_1(\boldsymbol{\tau}, \alpha) \neq m_3(\boldsymbol{\tau}, \alpha)$, where $m_1(\boldsymbol{\tau}, \alpha)$ and $m_3(\boldsymbol{\tau}, \alpha)$ are given by (2.27) and (2.28), respectively. Then, by Lemma A.2.1, the general solution to (2.24a) is
$$\mathbf{z}(x_3) = (a\mathbf{t}_1 + b\mathbf{t}_2)\, e^{m_1(\boldsymbol{\tau},\alpha)x_3} + c\mathbf{t}_3 e^{m_3(\boldsymbol{\tau},\alpha)x_3}, \tag{2.38}$$
where
$$\mathbf{t}_1 = \begin{pmatrix} \frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} \\ 0 \\ -\mathrm{i}\frac{\tau_1}{\lambda_1} \end{pmatrix}, \quad \mathbf{t}_2 = \begin{pmatrix} 0 \\ \frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} \\ -\mathrm{i}\frac{\tau_2}{\lambda_2} \end{pmatrix}, \quad \mathbf{t}_3 = \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ \frac{m_3(\boldsymbol{\tau},\alpha)}{\lambda_3} \end{pmatrix}. \tag{2.39}$$

We must also satisfy the boundary conditions (2.24b), so

$$
\begin{aligned}
\mathbf{0} &= \mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x_3}(0) + \mathbf{N}\mathbf{z}(0) \\
&= a(m_1(\boldsymbol{\tau},\alpha)\mathbf{M}+\mathbf{N})\mathbf{t}_1 + b(m_1(\boldsymbol{\tau},\alpha)\mathbf{M}+\mathbf{N})\mathbf{t}_2 + c(m_3(\boldsymbol{\tau},\alpha)\mathbf{M}+\mathbf{N})\mathbf{t}_3 \\
&= a\begin{pmatrix} \lambda_3\left(\frac{m_1(\boldsymbol{\tau},\alpha)^2}{\lambda_3^2}+\frac{\tau_1^2}{\lambda_1^2}\right) \\ \lambda_3\frac{\tau_1\tau_2}{\lambda_1\lambda_2} \\ -2\mathrm{i}\frac{\tau_1}{\lambda_1}m_1(\boldsymbol{\tau},\alpha) \end{pmatrix} + b\begin{pmatrix} \lambda_3\frac{\tau_1\tau_2}{\lambda_1\lambda_2} \\ \lambda_3\left(\frac{m_1(\boldsymbol{\tau},\alpha)^2}{\lambda_3^2}+\frac{\tau_2^2}{\lambda_2^2}\right) \\ -2\mathrm{i}\frac{\tau_2}{\lambda_2}m_1(\boldsymbol{\tau},\alpha) \end{pmatrix} \\
&\quad + c\begin{pmatrix} 2\mathrm{i}\frac{\tau_1}{\lambda_1}m_3(\boldsymbol{\tau},\alpha) \\ 2\mathrm{i}\frac{\tau_2}{\lambda_2}m_3(\boldsymbol{\tau},\alpha) \\ \frac{1}{\lambda_3}\left(\alpha^2+\left(\frac{\lambda_3^2}{\lambda_1^2}+1\right)\tau_1^2+\left(\frac{\lambda_3^2}{\lambda_2^2}+1\right)\tau_2^2\right) \end{pmatrix} \\
&= \lambda_3\begin{pmatrix} \frac{m_1(\boldsymbol{\tau},\alpha)^2}{\lambda_3^2}+\frac{\tau_1^2}{\lambda_1^2} & \frac{\tau_1\tau_2}{\lambda_1\lambda_2} & 2\mathrm{i}\frac{\tau_1}{\lambda_1}\frac{m_3(\boldsymbol{\tau},\alpha)}{\lambda_3} \\ \frac{\tau_1\tau_2}{\lambda_1\lambda_2} & \frac{m_1(\boldsymbol{\tau},\alpha)^2}{\lambda_3^2}+\frac{\tau_2^2}{\lambda_2^2} & 2\mathrm{i}\frac{\tau_2}{\lambda_2}\frac{m_3(\boldsymbol{\tau},\alpha)}{\lambda_3} \\ -2\mathrm{i}\frac{\tau_1}{\lambda_1}\frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} & -2\mathrm{i}\frac{\tau_2}{\lambda_2}\frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} & \frac{m_1(\boldsymbol{\tau},\alpha)^2}{\lambda_3^2}+\frac{\tau_1^2}{\lambda_1^2}+\frac{\tau_2^2}{\lambda_2^2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix},
\end{aligned}
$$

necessitating that the determinant of the matrix in the last line vanishes. This is if and only if

$$
\left(\frac{\tau_1^2}{\lambda_1^2}+\frac{\tau_2^2}{\lambda_2^2}+\frac{m_1(\boldsymbol{\tau},\alpha)^2}{\lambda_3^2}\right)^2 - \frac{4m_1(\boldsymbol{\tau},\alpha)m_3(\boldsymbol{\tau},\alpha)}{\lambda_3^2}\left(\frac{\tau_1^2}{\lambda_1^2}+\frac{\tau_2^2}{\lambda_2^2}\right) = 0. \tag{2.40}
$$

Dividing through by $|\boldsymbol{\tau}|^4$ leaves terms which only depend on $\frac{\tau_1}{|\boldsymbol{\tau}|}$ and $\frac{\tau_2}{|\boldsymbol{\tau}|}$, so without loss of generality, we henceforth assume (in this case) that $\boldsymbol{\tau} \in \mathbb{S}^1$. Hence, $m_1(\boldsymbol{\tau},\alpha)$ and $m_3(\boldsymbol{\tau},\alpha)$ simplify to $\widetilde{m}_1(\alpha)$ and $\widetilde{m}_3(\boldsymbol{\tau},\alpha)$, given by (2.34) and (2.35), respectively. Therefore, there exists a nontrivial solution to (2.24) for some $\boldsymbol{\tau} \neq \mathbf{0}$ if and only if

$$
\left(\frac{\tau_1^2}{\lambda_1^2}+\frac{\tau_2^2}{\lambda_2^2}+\frac{\tilde{m}_1(\alpha)^2}{\lambda_3^2}\right)^2 - \frac{4\tilde{m}_1(\alpha)\tilde{m}_3(\boldsymbol{\tau},\alpha)}{\lambda_3^2}\left(\frac{\tau_1^2}{\lambda_1^2}+\frac{\tau_2^2}{\lambda_2^2}\right) = 0. \tag{2.41}
$$

Degenerate case Now suppose α and $\boldsymbol{\tau}$ are such that $m_1(\boldsymbol{\tau},\alpha) = m_3(\boldsymbol{\tau},\alpha)$. By Lemma A.2.2, the general solution to (2.24a) is of the form

$$
\mathbf{z}(x_3) = \left[\mathbf{A} - \frac{(\lambda_1\lambda_2\lambda_3)^2 s\mathbf{t}_3\cdot\mathbf{A}}{(2+\lambda_1^2\lambda_2^2 s)m_1(\boldsymbol{\tau},\alpha)}x_3\mathbf{t}_3\right]e^{m_1(\boldsymbol{\tau},\alpha)x_3}, \tag{2.42}
$$

where $\mathbf{A} \in \mathbb{C}^3$ is an arbitrary constant vector, and $\mathbf{t}_3$ is given by $(2.39)_3$. We now check for particular solutions that satisfy (2.24b),

$$
\begin{aligned}
0 &= \mathbf{M}\mathbf{z}'(0) + \mathbf{N}\mathbf{z}(0) \\
&= (m_1(\boldsymbol{\tau},\alpha)\mathbf{M} + \mathbf{N})\mathbf{A} - \frac{(\lambda_1\lambda_2\lambda_3)^2 s\ \mathbf{t}_3 \cdot \mathbf{A}}{(2+\lambda_1^2\lambda_2^2 s)m_1(\boldsymbol{\tau},\alpha)} \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ (1+\lambda_1^2\lambda_2^2 s)\frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} \end{pmatrix} \\
&= \left[m_1(\boldsymbol{\tau},\alpha)\mathbf{M} + \mathbf{N} - \frac{(\lambda_1\lambda_2\lambda_3)^2 s}{(2+\lambda_1^2\lambda_2^2 s)m_1(\boldsymbol{\tau},\alpha)} \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ (1+\lambda_1^2\lambda_2^2 s)\frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} \end{pmatrix} \otimes \mathbf{t}_3 \right] \mathbf{A}.
\end{aligned}
$$

Hence, for there to exist nonzero $\mathbf{A}$ which satisfy (2.24b), we require that

$$
\det \left[m_1(\boldsymbol{\tau},\alpha)\mathbf{M} + \mathbf{N} - \frac{(\lambda_1\lambda_2\lambda_3)^2 s}{(2+\lambda_1^2\lambda_2^2 s)m_1(\boldsymbol{\tau},\alpha)} \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ (1+\lambda_1^2\lambda_2^2 s)\frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} \end{pmatrix} \otimes \mathbf{t}_3 \right] = 0. \tag{2.43}
$$

Note that

$$
m_1(\boldsymbol{\tau},\alpha)\mathbf{M} + \mathbf{N} = \begin{pmatrix} m_1(\boldsymbol{\tau},\alpha) & 0 & \mathrm{i}\lambda_3\frac{\tau_1}{\lambda_1} \\ 0 & m_1(\boldsymbol{\tau},\alpha) & \mathrm{i}\lambda_3\frac{\tau_2}{\lambda_2} \\ \mathrm{i}\lambda_3\gamma\frac{\tau_1}{\lambda_1} & \mathrm{i}\lambda_3\gamma\frac{\tau_2}{\lambda_2} & (1+\lambda_1^2\lambda_2^2 s)m_1(\boldsymbol{\tau},\alpha) \end{pmatrix},
$$

where $\gamma = -1 + \lambda_1^2\lambda_2^2 s$, and in particular, note that the third column of $m_1(\boldsymbol{\tau},\alpha)\mathbf{M} + \mathbf{N}$ is a multiple of the first vector in the tensor product appearing in (2.43). Therefore, this rank-one matrix does not contribute to the determinant evaluated in (2.43). Hence (2.43) holds if and only if

$$
\begin{aligned}
0 &= \det(m_1(\boldsymbol{\tau},\alpha)\mathbf{M} + \mathbf{N}) \\
&= (\lambda_1\lambda_2\lambda_3)^2 s \frac{m_1(\boldsymbol{\tau},\alpha)^3}{\lambda_3^2},
\end{aligned}
$$

which is not possible since $m_1(\boldsymbol{\tau},\alpha) \neq 0$. Hence, *there do not exist nontrivial solutions to* (2.24) *in the case* $m_1(\boldsymbol{\tau},\alpha) = m_3(\boldsymbol{\tau},\alpha)$.

Overall, we have the following two cases.

The complementing condition In the case $\alpha = 0$, $\widetilde{m}_1(0) = 1$, so we have that the complementing condition fails if and only if there exists $\boldsymbol{\tau} \in \mathbb{S}^1$ such that $\widetilde{m}_3(\boldsymbol{\tau},0) \neq 1$ and (2.41) holds (since the degenerate case $\widetilde{m}_3(\boldsymbol{\tau},0) = 1$ never permits nontrivial solutions). We note that $\widetilde{m}_3(\boldsymbol{\tau},0) = 1$ *for all* $\boldsymbol{\tau} \in \mathbb{S}^1$ if and only if $\lambda_1 = \lambda_2 = \lambda_3$. Therefore, the pair $(\mathbf{C}, \mathbf{n})$ satisfies the complementing condition if and only if $\lambda_1 = \lambda_2 = \lambda_3$, or if (2.36) holds for all $\boldsymbol{\tau} \in \mathbb{S}^1$ such that $\widetilde{m}_3(\boldsymbol{\tau}, 0) \neq 1$.

Agmon's Condition We have that Agmon's condition fails if and only if there exist $\alpha > 0$ and $\boldsymbol{\tau} \in \mathbb{S}^1$ such that $\widetilde{m}_1(\alpha) \neq \widetilde{m}_3(\boldsymbol{\tau}, \alpha)$ and (2.41) holds (since the degenerate case $\widetilde{m}_1(\alpha) = \widetilde{m}_3(\boldsymbol{\tau}, \alpha)$ never permits nontrivial solutions). Note that we cannot have $\widetilde{m}_3(\boldsymbol{\tau}, \alpha) = \widetilde{m}_1(\boldsymbol{\tau}, \alpha)$ *for all* $\alpha > 0$ and $\boldsymbol{\tau} \in \mathbb{S}^1$. Therefore, the pair $(\mathbf{C}, \mathbf{n})$ satisfies Agmon's condition if and only if (2.37) holds for all $\boldsymbol{\tau} \in \mathbb{S}^1$ and $\alpha > 0$ such that $\widetilde{m}_1(\alpha) \neq \widetilde{m}_3(\boldsymbol{\tau}, \alpha)$.

□

Remark 2.1.16. Usmani and Beatty [UB74] study instabilities of a neo-Hookean material occupying the three-dimensional half-space subjected to the pure homogeneous deformation $\boldsymbol{\varphi}^h$ given by (2.1), effectively checking the complementing condition. Their criterion for instability, [UB74, equation (3.16)] (i.e. when the complementing condition fails), is the converse of (2.36). However, their proof is incomplete as they do not consider the case $\widetilde{m}_3(\boldsymbol{\tau}, 0) = 1$, which is degenerate and corresponds to $\widetilde{m}_3(\boldsymbol{\tau}, 0) = m_1(\boldsymbol{\tau}, 0)$.[8] For further details, see Lemma A.2.2, where the general solution (2.42) for the degenerate case is derived.

Theorem 2.1.15 allows a much simpler verification of the complementing condition or Agmon's condition for stored energy functions of the form (2.25), by simply checking (2.36) or (2.37), respectively. However, unlike the two dimensional case with (2.11) and (2.12), the conditions (2.36) and (2.37) still depend on (varying) α and $\boldsymbol{\tau}$. It is unknown whether we need stricter assumptions on H to eliminate α and $\boldsymbol{\tau}$ from these conditions.

2.2 Obtaining incompressible results

In this section, we consider a one-parameter family of isotropic stored energy functions $\Phi(\lambda_1, \ldots, \lambda_n, k)$ depending on an 'incompressibility parameter' k such that $\Phi(\lambda_1, \ldots, \lambda_n, k) \to \infty$ as $k \to \infty$ if $\lambda_1 \ldots \lambda_n \neq 1$ but remains bounded if $\lambda_1 \ldots \lambda_n = 1$. We will obtain the corresponding results for incompressible materials in Chapter 3 from our results for compressible materials obtained in the previous section, with corresponding isotropic stored energy function $\Phi(\lambda_1, \ldots \lambda_n, k)$, by taking this 'incompressible limit' $k \to \infty$.

[8] Usmani and Beatty first take an incompressible limit before excluding the case of a repeated root: their allowance of the repeated root results only in trivial solutions through the false consequence that the general solution is still of the nondegenerate form (2.38); see [UB74] for further details.

2.2.1 The two dimensional case

Compressible neo-Hookean stored energy functions

Consider the stored energy function given by (2.13), but with $H(\cdot) = H(\cdot, k)$ also being a function of the parameter $k \geq 1$.

Simpson and Spector ([SS08a, §8, equation (8.1)]) studied a related example,[9] with the choice $H(d,k) = \frac{1}{k} d^{-k}$, in the limit $k \to \infty$. For this example they obtain that Agmon's condition is satisfied for $\lambda_1 \geq \lambda_\infty$ in the limit $k \to \infty$, where $\lambda_\infty \approx 0.544$ is Biot's critical compression ratio for the incompressible case [Bio63, equation (4.2)], as expected for an incompressible limit. However, given a deformation such that $\det(\nabla \boldsymbol{\varphi}) \geq 1$, we have that $\lim\limits_{k\to\infty} H(\det(\nabla \boldsymbol{\varphi}), k)$ remains finite, so this choice of H only restricts $\det(\nabla \boldsymbol{\varphi}) \geq 1$ in the limit $k \to \infty$. It is arguable that one should obtain the restriction $\det(\nabla \boldsymbol{\varphi}) = 1$ in the limit $k \to \infty$.

An arguably more suitable choice of H with which to seek an incompressible limit is the convex function $H : (0, \infty) \times [1, \infty) \to \mathbb{R}$ defined by

$$H(d,k) = \frac{2}{k} d^{-k} + \frac{1}{k} d^k. \tag{2.44}$$

Note that:

- For any $k \geq 1$, we have that $H(1,k) = \frac{3}{k}$.
- The traction free boundary condition for the identity deformation, equivalent to $H'(1,k) = -1$, is satisfied for all $k \geq 1$.
- In the limit $k \to \infty$, we have
  - $H(1,k) \to 0$,
  - for any $d \neq 1$, $H(d,k) \to \infty$.
- For any λ_1 and λ_2, by (2.16),

$$s = H''(\lambda_1\lambda_2, k) = 2(k+1)(\lambda_1\lambda_2)^{-k-2} + (k-1)(\lambda_1\lambda_2)^{k-2} \to \infty \qquad \text{as } k \to \infty.$$

Then an application of this limit restricts $\lambda_1\lambda_2 = 1$ (since if $\lambda_1\lambda_2 \neq 1$, the stored energy tends to infinity, since $H(\lambda_1\lambda_2, k) \to \infty$ in the limit $k \to \infty$). Furthermore, applying the limit $k \to \infty$ to (2.15) divided by s implies that the pair $(\mathbf{C}, \mathbf{n})$ satisfies Agmon's condition if and only if

$$\widetilde{F}\left(\frac{\lambda_2}{\lambda_1}\right) \leq 0, \tag{2.45}$$

[9] Also used in [NMMP12, equation (4.6)]

where

$$\widetilde{F}(r) = r^3 - 3r^2 - r - 1. \tag{2.46}$$

The cubic polynomial $\widetilde{F}(r)$ has one real root

$$r^* \approx 3.383, \tag{2.47}$$

and is negative for all $r < r^*$ (see Figure 2-2). Hence by (2.45), Agmon's condition is satisfied if and only if

$$\frac{\lambda_2}{\lambda_1} \leq r^*.$$

The critical stretch ratio $\frac{\lambda_2}{\lambda_1} = r^*$ coincides with $\sqrt{\frac{\lambda_1}{\lambda_2}} \approx 0.544$, agreeing with Biot instability [Bio63, equation (4.3)]; an incompressible, neo-Hookean setting.

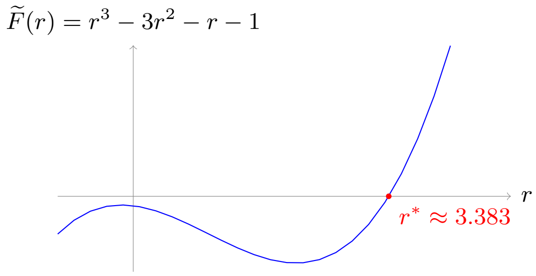


Figure 2-2: The cubic polynomial in $\widetilde{F}$ that has one real root.

Note that there are many alternatives to the choice of H, but our argument that yields this result is that s becomes very large as we reduce compressibility (by increasing $H(d,k)$ for $d \neq 1$ as k increases).

Remark 2.2.1. We have neglected to include the traction free boundary condition (2.14) in this incompressible limit for the following reason. The satisfaction of the traction boundary condition changes in the incompressible setting with the inclusion of an arbitrary pressure function p corresponding to a Lagrange multiplier. Indeed, we can see this in the present case by adding an 'arbitrary' term pd to our definition of H in (2.44), so that

$$H(d,k) = pd + \frac{2}{k}d^{-k} + \frac{1}{k}d^k.$$

This leaves the implications of our incompressible limit unchanged except for the change in the boundary condition (2.14), where we now add p to the left hand side (see also Remark 1.2.3).

General isotropic stored energy functions

Consider the isotropic stored energy function Φ given by

$$\Phi(\lambda_1, \lambda_2, k) = \Phi^{\mathrm{inc}}(\lambda_1, \lambda_2) + H(\lambda_1\lambda_2, k), \tag{2.48}$$

where $\Phi^{\mathrm{inc}} : (0, \infty)^2 \to \mathbb{R}$ satisfies the strong ellipticity inequalities (2.10), and $H : (0, \infty) \times [1, \infty) \to (0, \infty)$ is some function that is convex and twice-differentiable in its first argument, and continuous in its second argument.

Remark 2.2.2. By an elementary verification of equations (2.10) for the function $\Phi(\lambda_1, \lambda_2, k)$, if $s = H''(\lambda_1\lambda_2, k)$ is sufficiently large, then $\mathbf{C}$ is strongly elliptic.

Therefore, by Proposition 2.1.5, for sufficiently large s, the pair $(\mathbf{C}, \mathbf{n})$ satisfies Agmon's Condition if and only if

$$\begin{aligned}(\Phi_{11}^{\mathrm{inc}}+\lambda_2^2 s)(\Phi_{22}^{\mathrm{inc}} + \lambda_1^2 s) - \left(\Phi_{12}^{\mathrm{inc}} - \frac{\Phi_2^{\mathrm{inc}}}{\lambda_1} + \lambda_1\lambda_2 s\right)^2 \\ + \Psi_{12}^{\mathrm{inc}}\left(1 - \frac{\lambda_2^2}{\lambda_1^2}\right)\sqrt{(\Phi_{11}^{\mathrm{inc}} + \lambda_2^2 s)(\Phi_{22}^{\mathrm{inc}} + \lambda_1^2 s)} \geq 0.\end{aligned} \tag{2.49}$$

Suppose H is given by (2.44). In the limit $k \to \infty$, we have that $s \to \infty$ together with the restriction $\lambda_1\lambda_2 = 1$. Then by taking (2.49) divided by s, in the limit $k \to \infty$, we obtain that the pair $(\mathbf{C}, \mathbf{n})$ satisfies Agmon's condition if and only if

$$\lambda_1^2\Phi_{11}^{\mathrm{inc}} - 2\Phi_{12}^{\mathrm{inc}} + \frac{1}{\lambda_1^2}\Phi_{22}^{\mathrm{inc}} + \frac{\Phi_1^{\mathrm{inc}}}{\lambda_1} + \left(\frac{2}{\lambda_1} - \frac{1}{\lambda_1^3}\right)\Phi_2^{\mathrm{inc}} \geq 0. \tag{2.50}$$

Remark 2.2.3. We repeat the sentiment in Remark 2.2.1 with respect to the neglection of the boundary condition (2.9).

We will verify this agrees with the corresponding incompressible result in Chapter 3, see Theorem 3.3.4. In particular, if Φ^{inc} is of the form $\Phi^{\mathrm{inc}}(\lambda_1, \lambda_2) = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right)$, we have that (2.50) simplifies to (2.45), as expected.

2.2.2 The three dimensional case

Suppose that Φ is of the form (2.25), with $H(\cdot) = H(\cdot, k)$ now also being a function of the parameter $k \geq 1$. In particular, suppose that H is of the form (2.44). Similar to the case in two dimensions, we can take an incompressible limit by letting $k \to \infty$. This gives the limit $s \to \infty$, where s is given by (2.29), and we restrict $\lambda_1\lambda_2\lambda_3 = 1$.

Furthermore, by (2.35), we have $\widetilde{m}_3(\boldsymbol{\tau},\alpha) \to \sigma(\boldsymbol{\tau})$, where

$$\sigma(\boldsymbol{\tau}) := \frac{1}{\lambda_1\lambda_2}\sqrt{\frac{\tau_1^2}{\lambda_1^2}+\frac{\tau_2^2}{\lambda_2^2}}.$$

Hence, in the limit $k \to \infty$, (2.36) holds if and only if

$$(1-\sigma(\boldsymbol{\tau}))\widetilde{F}(\sigma(\boldsymbol{\tau})) \neq 0 \tag{2.51}$$

for all $\boldsymbol{\tau} \in \mathbb{S}^1$ such that $\sigma(\boldsymbol{\tau}) \neq 1$, and (2.37) holds if and only if

$$\left(1-\frac{\sigma(\boldsymbol{\tau})}{\widetilde{m}_1(\alpha)}\right)\widetilde{F}\left(\frac{\sigma(\boldsymbol{\tau})}{\widetilde{m}_1(\alpha)}\right) \neq 0 \tag{2.52}$$

for all $\alpha > 0$ and $\boldsymbol{\tau} \in \mathbb{S}^1$ such that $\sigma(\boldsymbol{\tau}) \neq \widetilde{m}_1(\alpha)$, where $\widetilde{m}_1(\alpha)$ is given by (2.27) and $\widetilde{F}$ is given by (2.46). In particular, $\widetilde{F}(r) \neq 0$ if and only if $r \neq r^*$, where r^* is given by (2.47). Hence, (2.51) holds if and only if

$$\sigma(\boldsymbol{\tau}) \neq r^*, \tag{2.53}$$

for all $\boldsymbol{\tau} \in \mathbb{S}^1$, and (2.52) holds if and only if

$$\frac{\sigma(\boldsymbol{\tau})}{\widetilde{m}_1(\alpha)} \neq r^* \tag{2.54}$$

for all $\alpha > 0$ and $\boldsymbol{\tau} \in \mathbb{S}^1$. Note that $\sigma(\boldsymbol{\tau})$ satisfies

$$\min\left\{\frac{1}{\lambda_1^2\lambda_2},\frac{1}{\lambda_1\lambda_2^2}\right\} = \min_{|\boldsymbol{\tau}|=1}\{\sigma(\boldsymbol{\tau})\} \leq \sigma(\boldsymbol{\tau}) \leq \max_{|\boldsymbol{\tau}|=1}\{\sigma(\boldsymbol{\tau})\} = \max\left\{\frac{1}{\lambda_1^2\lambda_2},\frac{1}{\lambda_1\lambda_2^2}\right\}.$$

Hence, overall, by (2.53), the pair $(\mathbf{C},\mathbf{n})$ satisfies the complementing condition if and only if

$$\max\left\{\frac{1}{\lambda_1^2\lambda_2},\frac{1}{\lambda_1\lambda_2^2}\right\} < r^* \quad \text{or} \quad \min\left\{\frac{1}{\lambda_1^2\lambda_2},\frac{1}{\lambda_1\lambda_2^2}\right\} > r^*. \tag{2.55}$$

Similarly, since the left-hand-side of (2.54) is decreasing (to 0) as α increases (to ∞), the pair $(\mathbf{C},\mathbf{n})$ satisfies Agmon's condition if and only if

$$\max\left\{\frac{1}{\lambda_1^2\lambda_2},\frac{1}{\lambda_1\lambda_2^2}\right\} \leq r^*. \tag{2.56}$$

Nowinski in [Now69, equation (28) or (29)] seems to be the first to obtain an incompressible instability criterion of the half-space for general $\lambda_1,\lambda_2 > 0$, equivalent to the complement of (2.51). The simplification of his criterion to (2.55) was not shown for general $\lambda_1, \lambda_2 > 0$, but after further analysis, Nowinski illustrated the region for which $(\lambda_1, \lambda_2) \in (0,1)^2$ satisfy this instability criterion;[10] see [Now69, Fig. 2]. Usmani and Beatty obtained the same criterion for instability as Nowinski [UB74, equation (3.22) or (3.23)] by taking an incompressible limit in a similar method to the present work (namely the limit $s \to \infty$), and illustrated the region of instability for all $\lambda_1, \lambda_2 > 0$; see [UB74, Figure 3]. We remark that the 'instability analysis' by Usmani and Beatty happens to be identical to verifying the failure of the complementing condition, though this is not mentioned in [UB74]. We have included a similar diagram in Figure 2-3 derived explicitly from (2.55).

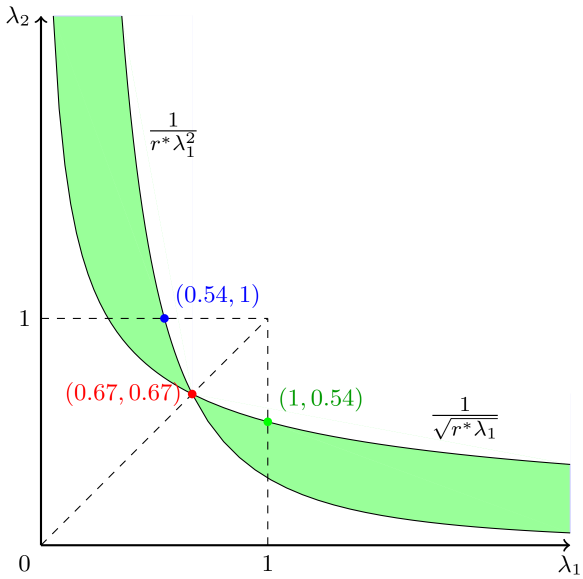


Figure 2-3: The region for failure of the complementing condition is shaded green. Points on the boundary of this region also count for failure. See equation (2.55).

The region for failure of Agmon's Condition, given by the complement of (2.56), agrees with the "region of instability" from Chen et al [CYW18, equation (6.31)]. The range of values for which λ_1, λ_2 satisfy (2.56) is well illustrated by [CYW18, Fig. 2], but we have included a similar diagram that illustrates the jump in gradient at the point $(0.67, 0.67)$ in greater detail; see Figure 2-4.

Setting either $\lambda_1 = 1$ or $\lambda_2 = 1$ results in the two dimensional case where Agmon's condition holds if and only if $\frac{\lambda_3}{\lambda_2} \leq r^*$ or $\frac{\lambda_3}{\lambda_1} \leq r^*$, respectively, where r^* is given by (2.47). This is due to the fact that setting either λ_1 or λ_2 to 1 would yield the

[10]Nowinski seems to have obtained the diagram [Now69, Fig. 2] directly from his instability criterion [Now69, equation (29)].

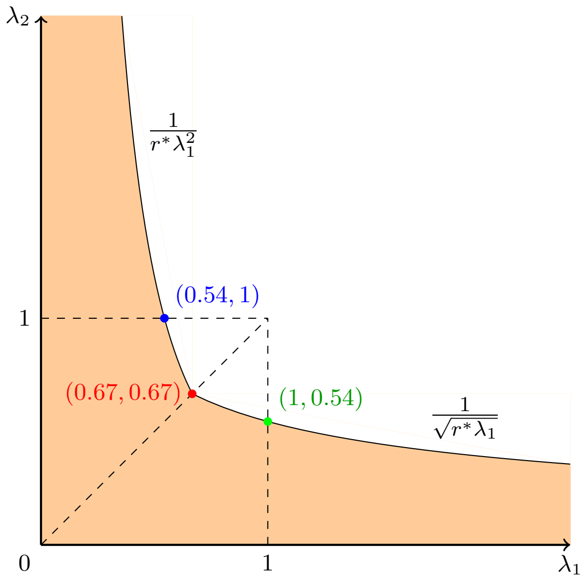


Figure 2-4: The region for failure of Agmon's condition is shaded orange. Points on the boundary of this region do not cause failure.

respective two-dimensional problem in the cross-section, and with zero change in the x_1 or x_2 direction, respectively. Note that $\sqrt{\frac{1}{r^*}} \approx 0.544$ is Biot's critical compression ratio [Bio63, equation (4.3)].

Chapter 3

Necessary conditions for weak local minimisers in incompressible hyperelasticity

Let $\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in M_1^{n\times n}$, and let $\boldsymbol{\varphi}^h = \mathbf{Dx}$. Our main goal in this chapter is analogous to that of Chapter 2: we aim to find a concise algebraic condition depending only on Φ^{inc} and the principal stretches $\lambda_1, \ldots, \lambda_n$ which holds if and only if the incompressible version of Agmon's condition holds (as stated in Theorem 1.2.38 (2), with $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$).

Define, for $\mathbf{F} \in M_1^{n\times n}$,

$$\mathcal{T}(\mathbf{F}) = \{\mathbf{G} \in M^{n\times n} \mid \operatorname{Cof}\mathbf{F} \cdot \mathbf{G} = 0\}, \tag{3.1}$$

and for $\mathbf{F} \in M_1^{n\times n}$, $\mathbf{G} \in \mathcal{T}(\mathbf{F})$,

$$\mathcal{N}(\mathbf{F}, \mathbf{G}) = \{\mathbf{K} \in M^{n\times n} \mid \operatorname{Cof}'(\mathbf{F})[\mathbf{G}, \mathbf{G}] + \operatorname{Cof}\mathbf{D} \cdot \mathbf{K} = 0\}, \tag{3.2}$$

where

$$\begin{aligned} \operatorname{Cof}'(\mathbf{F})[\mathbf{G}, \mathbf{G}] &:= \left(\frac{\partial^2}{\partial \mathbf{F}^2} \det(\mathbf{F})\right)[\mathbf{G}, \mathbf{G}] \\ &= \begin{cases} 2\det(\mathbf{G}) & \text{if } n = 2, \\ 2\operatorname{Cof}\mathbf{G} \cdot \mathbf{F} & \text{if } n = 3. \end{cases} \end{aligned} \tag{3.3}$$

If $W^{\text{inc}} \in C^2(M_1^{n\times n}, \mathbb{R}^n)$, then the first and second gradients of W^{inc} at any $\mathbf{F} \in M_1^{n\times n}$, denoted by $\nabla W^{\text{inc}}(\mathbf{F})$ and $\nabla^2 W^{\text{inc}}(\mathbf{F})$ respectively, are linear and bilinear operators[1]

$$\begin{aligned}\nabla W^{\text{inc}}(\mathbf{F}) &: \mathcal{T}(\mathbf{F}) \to \mathbb{R},\\ \nabla^2 W^{\text{inc}}(\mathbf{F}) &: \mathcal{T}(\mathbf{F}) \times \mathcal{T}(\mathbf{F}) \to \mathbb{R}.\end{aligned}$$

However, calculating partial derivatives of W^{inc}, for example $\frac{\partial W^{\text{inc}}(\mathbf{F})}{\partial F_{i\alpha}}$, implicitly involves using an *extension* of W^{inc}: any incompressible, frame-indifferent, homogeneous stored energy function $W^{\text{inc}} : M_1^{n\times n} \to \mathbb{R}$ may be extended to a frame indifferent, homogeneous stored energy function $W : M_+^{n\times n} \to \mathbb{R}$, satisfying

$$W^{\text{inc}}(\mathbf{F}) = W(\mathbf{F}), \quad \text{for all } \mathbf{F} \in M_1^{n\times n}. \tag{3.4}$$

For example, one may choose

$$W(\mathbf{F}) = W^{\text{inc}}\left(\frac{\mathbf{F}}{(\det \mathbf{F})^{\frac{1}{n}}}\right), \qquad \mathbf{F} \in M_+^{n\times n}.$$

If $W^{\text{inc}} \in C^2(M_1^{n\times n}, \mathbb{R}^n)$, then $W \in C^2(M_+^{n\times n}, \mathbb{R})$. As is seemingly a common practice in incompressible elasticity, from now on we will use a particular extension W satisfying (3.4) instead of W^{inc}.

Similarly, in the isotropic case, we may extend Φ^{inc} to a function Φ satisfying $\Phi(\boldsymbol{\lambda}) = \Phi^{\text{inc}}(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \Lambda_n$, where Λ_n is given by (1.9). The first and second gradients of Φ^{inc} at some $\boldsymbol{\lambda} \in \Lambda_n$, denoted $\nabla\Phi^{\text{inc}}(\boldsymbol{\lambda})$ and $\nabla^2\Phi^{\text{inc}}(\boldsymbol{\lambda})$, are linear and bilinear operators

$$\begin{aligned}\nabla\Phi^{\text{inc}}(\boldsymbol{\lambda}) &: \mathcal{T}(\boldsymbol{\lambda}) \to \mathbb{R},\\ \nabla^2\Phi^{\text{inc}}(\boldsymbol{\lambda}) &: \mathcal{T}(\boldsymbol{\lambda}) \times \mathcal{T}(\boldsymbol{\lambda}) \to \mathbb{R},\end{aligned}$$

where

$$\mathcal{T}(\boldsymbol{\lambda}) = \left\{ \mathbf{u} \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_n} \end{pmatrix} \cdot \mathbf{u} = 0 \right\}.$$

However, calculating partial derivatives such as $\Phi^{\text{inc}}_{,i}(\boldsymbol{\lambda})$ implicitly involves using some extension Φ satisfying $\Phi(\boldsymbol{\lambda}) = \Phi^{\text{inc}}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \Lambda_n$. Hence, our 'concise algebraic condition' for Agmon's condition must depend on some choice of extension Φ.

Note that Agmon's condition (for compressible elasticity, Definition 1.2.13) is ini-

[1] This observation that the derivatives are in fact linear or bilinear operators was also made by Fosdick and MacSithigh [FM86]

tially stated as a boundary value problem depending on the (compressible) elasticity tensor $\mathbf{C}$, given by (2.6). A key step in the proof of Propositions 2.1.4 and 2.1.5 in [Dav89, Theorem 8.1] and [MS98, Example 5.3], respectively[2] is that one can apply (2.8) to $\mathbf{C}$, and write it in terms of the (compressible) isotropic stored energy function Φ, its derivatives, and the stretches $\lambda_1, \ldots, \lambda_n$. This turns Agmon's condition (for compressible elasticity) into an algebraic condition on Φ, its derivatives, and the stretches $\lambda_1, \ldots, \lambda_n$ (equations (2.11) and (2.12)). We would like to make a similar simplification for Agmon's condition for incompressible elasticity, stated in Theorem 1.2.38 (2) as a boundary value problem depending on $\mathbb{K}$, where $\mathbb{K}$ is given by (1.50). Note that $\mathbb{K}$ depends on some extension W satisfying (3.4). Furthermore, since W^{inc} is isotropic, we may assume that W is isotropic, so that there exists a function Φ satisfying (1.8). Therefore, we may write $\mathbb{K}$ in terms of Φ, its derivatives, and the stretches $\lambda_1, \ldots, \lambda_n$ by applying Theorem 2.1.2. This lead to an algebraic condition for Agmon's condition as desired, but in terms of the extension Φ, and not Φ^{inc}. It is not a priori clear whether Agmon's condition depends on the choice of this extension Φ.

This chapter will be split into three sections. In the first section, we will show that a necessary condition for a weak local minimiser is that the functional J given by (1.49) must be nonnegative for all smooth solenoidal vector fields vanishing on ∂G_D. We will do this by defining the second variation $\delta^2 E$ in terms of the extended stored energy function W (see (3.8)), and applying Simpson and Spector's argument in [SS87, Proposition 4.2]. In the second section, we will show that $\delta^2 E$ and J are invariant with respect to the choice of extension W. In the third section, we give a direct derivation of those results in Section 2.2 which were obtained by taking an incompressible limit. Moreover, we will show that the result in the general isotropic case in two dimensions is invariant with respect to the choice of extension.

3.1 Necessary conditions at the boundary for weak local minimisers in incompressible hyperelasticity

For this section, all results also apply to stored energy functions which may *not* be isotropic.

MacSithigh [Mac05] shows that a necessary condition for a strong local minimiser $\boldsymbol{\varphi}$ is that the functional $J[\mathbf{w}]$ is nonnegative for all solenoidal $\mathbf{w}$ vanishing on ∂G_D (see Theorem 1.2.36). MacSithigh shows this by first showing that for any point $\mathbf{x}_0 \in \partial\Omega_T$ with deformed normal $\mathbf{N} = \mathbf{e}_n$, quasiconvexity at the boundary of W must hold at $(\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \mathbf{N})$ in the sense of Definition 1.2.31 (see Theorem 1.2.33), and then shows

[2] Or from Simpson and Spector [SS08a, Proposition 3.2 (ii) and (iii), respectively].

that this is a sufficient condition for the nonnegativity of J. The fact that this is also a necessary condition for a weak local minimiser is claimed in [Mac05] and [Mac07], but this is not a trivial corollary. In this section we will prove this claim (see Theorem 3.1.5) by using the same logic used to prove the analogous result in the compressible case, which is originally due to Simpson and Spector [SS89, Proposition 4.2] (see Remark 3.1.6). As a result, we will avoid the usage of (incompressible) quasiconvexity at the boundary as defined in Definition 1.2.31.

Firstly, we note the necessary preliminary results relating to weak local minimisers of

$$E[\boldsymbol{\varphi}] = \int_\Omega W(\nabla\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}\mathbf{x} - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x})\ \mathrm{d}S(\mathbf{x}),$$

each due to Fosdick and MacSithigh [FM86].

Theorem 3.1.1 (Fosdick and MacSithigh [FM86, Theorem 3.1])**.** *Let $\boldsymbol{\varphi} \in \mathcal{A}^{\mathrm{inc}}$ be a weak local minimiser of E, where $\mathcal{A}^{\mathrm{inc}}$ is given by* (1.42)*. Define the set of variations*

$$\mathcal{V} = \{\mathbf{u} \in C^1(\Omega, \mathbb{R}^n) \mid \nabla\mathbf{u}(\mathbf{x}) \in \mathcal{T}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\ \textit{for all}\ \mathbf{x} \in \Omega,\ \mathbf{u}(\mathbf{x}) = \mathbf{0}\ \textit{for all}\ \mathbf{x} \in \partial\Omega_D\}.$$

Then for all $\mathbf{u} \in \mathcal{V}$, and for all $\mathbf{v} \in C^1(\Omega, \mathbb{R}^n)$ such that $\nabla\mathbf{v}(\mathbf{x}) \in \mathcal{N}(\nabla\boldsymbol{\varphi}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))$ for all $\mathbf{x} \in \Omega$ and $\mathbf{v} = \mathbf{0}$ on $\partial\Omega_D$, we have

$$\int_\Omega \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}^2}[\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})] + \frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}}\cdot\nabla\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\ \mathrm{d}S(\mathbf{x}) \geq 0. \tag{3.5}$$

Theorem 3.1.2 (Fosdick and MacSithigh [FM86, Theorem 3.2])**.** *Let $\boldsymbol{\varphi} \in \mathcal{A}^{\mathrm{inc}}$ be a weak local minimiser. Then there exists a $p : \Omega \to \mathbb{R}^n$ such that*

$$\mathrm{Div}\left(\frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}}\right) = \mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x})\nabla p(\mathbf{x}), \quad \textit{for all}\ \mathbf{x} \in \Omega, \tag{3.6}$$

and

$$\left(\frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}} - p(\mathbf{x})\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x})\right)\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}), \quad \textit{for all}\ \mathbf{x} \in \partial\Omega_T. \tag{3.7}$$

Corollary 3.1.3 (Fosdick and MacSithigh [FM86, equation (3.24)])**.** *For a weak local minimiser $\boldsymbol{\varphi} \in \mathcal{A}^{\mathrm{inc}}$, define the* second variation *of E (at $\boldsymbol{\varphi}$) by*

$$\begin{aligned}\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] := \int_\Omega & \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}^2}[\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})] \\ & - p(\mathbf{x})\mathrm{Cof}'(\nabla\boldsymbol{\varphi}(\mathbf{x}))[\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})]\ \mathrm{d}\mathbf{x},\end{aligned} \tag{3.8}$$

where $p : \Omega \to \mathbb{R}^n$ *satisfies* (3.6) *and* (3.7). *Then*

$$\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] = \int_\Omega \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial \mathbf{F}^2}[\nabla\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x})] + \frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial \mathbf{F}} \cdot \nabla\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} \\ - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\ \mathrm{d}S(\mathbf{x})$$

for all $\mathbf{u} \in \mathcal{V}$ *and all* $\mathbf{v}$ *as in Theorem 3.1.1. Moreover,* (3.5) *holds for all* $\mathbf{u} \in \mathcal{V}$ *and all* $\mathbf{v}$ *as in Theorem 3.1.1 if and only if* $\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] \geq 0$ *for all* $\mathbf{u} \in \mathcal{V}$.

Remark 3.1.4. Corollary 3.1.3 is proved in [FM86, equation (3.24)] in $n = 3$ dimensions only, expressing the second term in a different, but equivalent, form.

We now present the key result in this section.

Theorem 3.1.5. *Let* $\boldsymbol{\varphi} \in \mathcal{A}^{\mathrm{inc}}$ *be a weak local minimiser. Then* $J[\mathbf{w}] \geq 0$ *for all solenoidal* $\mathbf{w} : G \to \mathbb{R}^n$ *such that* $\mathbf{w} = \mathbf{0}$ *on* ∂G_D*, where* J *is given by* (1.49).

Proof. By Corollary 3.1.3, $\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] \geq 0$ for all $\mathbf{u} \in \mathcal{V}$. That is, $\mathbf{0} \in \mathcal{V}$ is a global minimiser of the functional $\delta^2 E(\boldsymbol{\varphi})[\cdot]$ over the set $\mathcal{V}$. Hence, for all $\mathbf{x}_0 \in \partial\Omega_T$ with normal $\mathbf{n}$, the function $\widetilde{W}_0$ defined by

$$\widetilde{W}_0(\mathbf{G}) := \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))}{\partial \mathbf{F}^2}[\mathbf{G},\mathbf{G}] - p(\mathbf{x}_0)\mathrm{Cof}'(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\mathbf{G},\mathbf{G}], \quad \mathbf{G} \in \mathcal{T}(\nabla\boldsymbol{\varphi}(\mathbf{x}_0)) \tag{3.9}$$

is "quasiconvex at the boundary" at $(\mathbf{0}, \mathbf{n})$, in the (compressible) sense of Definition 1.2.21 with the constraint $\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0) \cdot \mathbf{G} = 0$. That is, for any standard boundary domain $D_\mathbf{n}$,

$$\int_{D_\mathbf{n}} \widetilde{W}_0(\nabla\boldsymbol{\psi}(\mathbf{x}))\ \mathrm{d}\mathbf{x} \geq 0, \tag{3.10}$$

for all $\boldsymbol{\psi} \in C^1(D_\mathbf{n}, \mathbb{R}^n)$ such that $\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0) \cdot \nabla\boldsymbol{\psi}(\mathbf{x}) = 0$ and $\boldsymbol{\psi} = \mathbf{0}$ on $\partial D_\mathbf{n} \setminus \Gamma_n$. This claim follows from a trivial modification of the proof of [BM84, Theorem 2.2], where variations $\boldsymbol{\psi}$ must satisfy the constraint $\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0) \cdot \nabla\boldsymbol{\psi}(\mathbf{x}) = 0$ (see Remark 1.2.23 and Remark 1.2.24). Hence, we have by (3.9) and (3.10) that

$$\int_{D_\mathbf{n}} \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))}{\partial \mathbf{F}^2}[\nabla\boldsymbol{\psi}(\mathbf{x}), \nabla\boldsymbol{\psi}(\mathbf{x})] \\ - p(\mathbf{x}_0)\mathrm{Cof}'(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\nabla\boldsymbol{\psi}(\mathbf{x}), \nabla\boldsymbol{\psi}(\mathbf{x})]\ \mathrm{d}\mathbf{x} \geq 0, \tag{3.11}$$

for all $\boldsymbol{\psi} \in C^1(D_\mathbf{n}, \mathbb{R}^n)$ such that $\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0) \cdot \nabla\boldsymbol{\psi}(\mathbf{x}) = 0$ and $\boldsymbol{\psi} = \mathbf{0}$ on $\partial D_\mathbf{n} \setminus \Gamma_n$. Transform to coordinates $\mathbf{y} = \nabla\boldsymbol{\varphi}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \boldsymbol{\varphi}(\mathbf{x}_0)$, and let $\mathbf{w}(\mathbf{y}) = \boldsymbol{\psi}(\mathbf{x})$. Define $\widetilde{D}_\mathbf{N}$ by

$$\widetilde{D}_\mathbf{N} = \{\mathbf{y} \in \mathbb{R}^n \mid \nabla\boldsymbol{\varphi}(\mathbf{x}_0)^{-1}(\mathbf{y} - \boldsymbol{\varphi}(\mathbf{x}_0)) + \mathbf{x}_0 \in D_\mathbf{n}\},$$

so that the normal to the deformed surface at $\mathbf{y}_0 = \boldsymbol{\varphi}(\mathbf{x}_0)$ is $\mathbf{N}(\mathbf{y}_0) = \mathbf{N}(\boldsymbol{\varphi}(\mathbf{x}_0))$, related to $\mathbf{n}(\mathbf{x}_0)$ by

$$\mathbf{N}(\boldsymbol{\varphi}(\mathbf{x}_0)) = \frac{\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0)\mathbf{n}(\mathbf{x}_0)}{|\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0)\mathbf{n}(\mathbf{x}_0)|}. \tag{3.12}$$

Then $\mathbf{w} \in C^1(\widetilde{D}_{\mathbf{N}}, \mathbb{R}^n)$ is solenoidal, and by (3.11),

$$\begin{aligned}\int_{\widetilde{D}_{\mathbf{N}}} &\frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))}{\partial \mathbf{F}^2}[\nabla\mathbf{w}(\mathbf{y})\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \nabla\mathbf{w}(\mathbf{y})\nabla\boldsymbol{\varphi}(\mathbf{x}_0)] \\ &- p(\mathbf{x}_0)\mathrm{Cof}'(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))[\nabla\mathbf{w}(\mathbf{y})\nabla\boldsymbol{\varphi}(\mathbf{x}_0), \nabla\mathbf{w}(\mathbf{y})\nabla\boldsymbol{\varphi}(\mathbf{x}_0)]\ \mathrm{d}\mathbf{y} \geq 0. \end{aligned} \tag{3.13}$$

We note that $p(\mathbf{x}_0)$ is such that (3.7) holds at $\mathbf{x} = \mathbf{x}_0$, so we have by (3.12) that

$$\begin{aligned}\left(\frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}_0))}{\partial \mathbf{F}}\nabla\boldsymbol{\varphi}(\mathbf{x}_0)^T - p(\mathbf{x}_0)\right)\mathbf{N}(\mathbf{y}_0) &= \frac{\mathbf{t}(\mathbf{x}_0)}{|\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0)\mathbf{n}(\mathbf{x}_0)|} \\ &= |\nabla\boldsymbol{\varphi}(\mathbf{x}_0)^T\mathbf{N}(\mathbf{y}_0)|\mathbf{t}(\mathbf{x}_0),\end{aligned}$$

so that $p(\mathbf{x}_0)$ satisfies (1.51). Furthermore, by writing $\mathbf{F}_0 = \nabla\boldsymbol{\varphi}(\mathbf{x}_0)$, since $\mathbf{w}$ is solenoidal, we have that

$$\begin{aligned}\mathrm{Cof}'(\mathbf{F}_0)[\nabla\mathbf{w}(\mathbf{y})\mathbf{F}_0, \nabla\mathbf{w}(\mathbf{y})\mathbf{F}_0] &= \begin{cases} 2\det(\nabla\mathbf{w}(\mathbf{y})\mathbf{F}_0) = 2\det(\nabla\mathbf{w}(\mathbf{y})) & n = 2, \\ 2\mathrm{Cof}(\nabla\mathbf{w}(\mathbf{y})\mathbf{F}_0)\cdot\mathbf{F}_0 = -\mathrm{tr}((\nabla\mathbf{w}(\mathbf{y}))^2) & n = 3. \end{cases} \\ &= -\mathrm{tr}((\nabla\mathbf{w}(\mathbf{y}))^2).\end{aligned}$$

Overall this gives, by (3.13), that

$$\int_{\widetilde{D}_{\mathbf{N}}} \mathbb{K}[\nabla\mathbf{w}(\mathbf{y}), \nabla\mathbf{w}(\mathbf{y})]\ \mathrm{d}\mathbf{y} \geq 0, \tag{3.14}$$

where $\mathbb{K}$ is given by (1.50). Finally, since G, given by (1.45), is a particular standard boundary domain with normal $\mathbf{N}(\mathbf{y}_0) = \mathbf{N} = \mathbf{e}_n$ (given by (1.44)), we have that $J[\mathbf{w}] \geq 0$ for all solenoidal $\mathbf{w}$ vanishing on ∂G_D. $\square$

Remark 3.1.6. The key steps in showing Theorem 3.1.5 are the same as for its compressible analogue in [SS89, Proposition 4.2]: we let $\boldsymbol{\varphi} \in \mathcal{A}^{\mathrm{inc}}$ be a weak local minimiser, so that the second variation is nonnegative. Therefore, the function $\mathbf{0}$ is a (strong) global minimiser for the second variation over the set $\mathcal{V}$, which implies that the quadratic functional $\widetilde{W_0}$ defined by (3.9) is quasiconvex at the boundary at $(\mathbf{0}, \mathbf{n})$ (with variations $\boldsymbol{\psi}$ satisfying the constraint $\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x}_0)\cdot\boldsymbol{\psi}(\mathbf{x}) = 0$). The remaining part of the proof shows that this holds if and only if (3.14) holds, and one can take the example standard boundary domain G to obtain the desired inequality central to Theorem 1.2.38 (which is the incompressible analogue of Theorem 1.2.26).

3.2 Extension invariance

Two important results in the previous section are Corollary 3.1.3 and Theorem 3.1.5, which give necessary conditions for a deformation $\boldsymbol{\varphi}$ to be weak local minimiser of the stored energy E^{inc}. These are, respectively, the nonnegativity of the second variation $\delta^2 E(\boldsymbol{\varphi})$ and the functional J. Both $\delta^2 E(\boldsymbol{\varphi})$ and J apparently depend on an extension W, hence it is important to verify that the choice of this extension leaves $\delta^2 E(\boldsymbol{\varphi})$ and J unchanged.

Lemma 3.2.1. *Let $\delta > 0$, and let $\mathbf{X} : (-\delta, \delta) \to M_1^{n\times n}$ be a one-parameter family of proper unimodular matrices (matrices with determinant 1), such that $\mathbf{X}(0) = \mathbf{F}$, where $\mathbf{F} \in M_1^{n\times n}$. Then*

$$\dot{\mathbf{X}}(0) \in \mathcal{T}(\mathbf{F}),$$

and

$$\ddot{\mathbf{X}}(0) \in \mathcal{N}(\mathbf{F}, \dot{\mathbf{X}}(0)),$$

where $\mathcal{T}(\mathbf{F})$ and $\mathcal{N}(\mathbf{F}, \dot{\mathbf{X}}(0))$ are given by (3.1) *and* (3.2)*, respectively.*

Proof. Differentiating the constraint $\det(\mathbf{X}(t)) = 1$ with respect to t once, then once more, gives

$$\begin{aligned}
&\operatorname{Cof}\mathbf{X}(t) \cdot \dot{\mathbf{X}}(t) = 0, \\
&\operatorname{Cof}'(\mathbf{X}(t))[\dot{\mathbf{X}}(t), \dot{\mathbf{X}}(t)] + \operatorname{Cof}\mathbf{X}(t) \cdot \ddot{\mathbf{X}}(t) = 0,
\end{aligned}$$

respectively. Setting $t = 0$ gives the result. □

Proposition 3.2.2. *Let $\widetilde{W}$ be any other extension of W^{inc}, so that $\widetilde{W}$ satisfies* (3.4)*. Then*

$$\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{G} = \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{G}, \qquad \textit{for all } \mathbf{G} \in \mathcal{T}(\mathbf{F}), \tag{3.15}$$

and

$$\begin{aligned}
\frac{\partial^2 W(\mathbf{F})}{\partial \mathbf{F}^2}[\mathbf{G}, \mathbf{G}] + \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{K} = \frac{\partial^2 \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}^2}[\mathbf{G}, \mathbf{G}] + \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{K}, \\
\textit{for all } \mathbf{G} \in \mathcal{T}(\mathbf{F}),\ \mathbf{K} \in \mathcal{N}(\mathbf{F}, \mathbf{G}).
\end{aligned} \tag{3.16}$$

Proof. Let $\delta > 0$, and let $\mathbf{X} : (-\delta, \delta) \to M_1^{n\times n}$ be a one-parameter family of proper unimodular matrices, such that $\mathbf{X}(0) = \mathbf{F}$. Then for all $t \in (-\delta, \delta)$, we have

$$W(\mathbf{X}(t)) = \widetilde{W}(\mathbf{X}(t)). \tag{3.17}$$

Then, by differentiating (3.17) with respect to t and setting $t = 0$, we have that

$$\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \cdot \dot{\mathbf{X}}(0) = \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \cdot \dot{\mathbf{X}}(0).$$

By Lemma 3.2.1, $\dot{\mathbf{X}}(0) \in \mathcal{T}(\mathbf{F})$, so the first claim follows. By differentiating (3.17) twice with respect to t and setting $t = 0$, we have that

$$\frac{\partial^2 W(\mathbf{F})}{\partial \mathbf{F}^2}[\dot{\mathbf{X}}(0), \dot{\mathbf{X}}(0)] + \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \cdot \ddot{\mathbf{X}}(0) = \frac{\partial^2 \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}^2}[\dot{\mathbf{X}}(0), \dot{\mathbf{X}}(0)] + \frac{\partial \widetilde{W}(\mathbf{F})}{\partial \mathbf{F}} \cdot \ddot{\mathbf{X}}(0).$$

By Lemma 3.2.1, $\ddot{\mathbf{X}}(0) \in \mathcal{N}(\mathbf{F}, \dot{\mathbf{X}}(0))$, so the second claim follows. $\square$

The following lemma is in fact a specific case of a remark by Fosdick and MacSithigh [FM86, Remark 3.1], in which a proof was omitted.

Lemma 3.2.3. *Let $\mathbf{F} \in M_1^{n\times n}$, and let $\mathbf{w} : G \to \mathbb{R}^n$ be solenoidal. Define $\mathbf{w}_1 = (\nabla \mathbf{w})\mathbf{w}$. Then*

$$\nabla \mathbf{w} \mathbf{F} \in \mathcal{T}(\mathbf{F}), \tag{3.18}$$

and

$$\nabla \mathbf{w}_1 \mathbf{F} \in \mathcal{N}(\mathbf{F}, \nabla \mathbf{w} \mathbf{F}). \tag{3.19}$$

Proof. Since $\mathbf{w}$ is solenoidal, (3.18) is trivial. Define the operator T by

$$T[\mathbf{w}] = \mathrm{Cof}'(\mathbf{F})[\nabla \mathbf{w}\mathbf{F}, \nabla \mathbf{w}\mathbf{F}] + \mathrm{Cof}\mathbf{F} \cdot \nabla((\nabla \mathbf{w})\mathbf{w}). \tag{3.20}$$

We will prove (3.19) by showing $T[\mathbf{w}] = 0$ in each case $n = 2$ and $n = 3$.

Case $\mathbf{n = 2}$ Note that by (3.3),

$$\mathrm{Cof}'(\mathbf{F})[\nabla \mathbf{w}\mathbf{F}, \nabla \mathbf{w}\mathbf{F}] = 2\det(\nabla \mathbf{w}\mathbf{F}) = 2\det \nabla \mathbf{w}, \tag{3.21}$$

since $\lambda_1 \lambda_2 = 1$. Since $\mathbf{w}$ is solenoidal, we also have

$$\mathrm{Cof}\mathbf{F} \cdot (\nabla \mathbf{w}_1 \mathbf{F}) = \mathrm{div}\,((\nabla \mathbf{w})\mathbf{w}) = \left(\frac{\partial w_1}{\partial x_1}\right)^2 + 2\frac{\partial w_1}{\partial x_2}\frac{\partial w_2}{\partial x_1} + \left(\frac{\partial w_2}{\partial x_2}\right)^2. \tag{3.22}$$

Hence by (3.21), (3.22), and the fact that $\mathbf{w}$ is solenoidal,

$$T[\mathbf{w}] = \left(\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}\right)^2 = 0.$$

Case $\mathbf{n} = 3$ In this case, by (3.3), and the fact that $\mathbf{w}$ is solenoidal, we have

$$\mathrm{Cof}'(\mathbf{F})[\nabla\mathbf{w}\mathbf{F}, \nabla\mathbf{w}\mathbf{F}] = \mathrm{Cof}(\nabla\mathbf{w}\mathbf{F}) \cdot \mathbf{F} = 2\mathrm{tr}(\mathrm{Cof}\nabla\mathbf{w}) = -\mathrm{tr}((\nabla\mathbf{w})^2), \quad (3.23)$$

and

$$\mathrm{Cof}\mathbf{F} \cdot (\nabla\mathbf{w}_1\mathbf{F}) = \mathrm{div}\,((\nabla\mathbf{w})\mathbf{w}) = \mathrm{tr}((\nabla\mathbf{w})^2).$$

Hence $T[\mathbf{w}] = 0$.

□

Proposition 3.2.4. *The following expressions are invariant with respect to the choice of extension W of W^{inc} from $M_1^{n\times n}$ to $M_+^{n\times n}$:*

(i) The second variation $\delta^2E(\boldsymbol{\varphi})[\mathbf{u}]$ for all $\mathbf{u} \in \mathcal{V}$, where $\boldsymbol{\varphi} \in \mathcal{A}^{\mathrm{inc}}$ is a weak local minimiser, and $\delta^2E(\boldsymbol{\varphi})$ is given by (3.8);

(ii) The functional $J[\mathbf{w}]$ for all solenoidal $\mathbf{w} : G \to \mathbb{R}^n$ such that $\mathbf{w} = \mathbf{0}$ on ∂G_D, where J is given by (1.49).

Proof of (i). Let $\widetilde{W}$ be any other extension of W^{inc} (in addition to W), and let $\delta^2\widetilde{E}(\boldsymbol{\varphi})$ denote the second variation corresponding to $\widetilde{W}$. By Corollary 3.1.3, for any $\mathbf{u} \in \mathcal{V}$ and all $\mathbf{v} : \Omega \to \mathbb{R}^n$ satisfying $\nabla\mathbf{v}(\mathbf{x}) \in \mathcal{N}(\nabla\boldsymbol{\varphi}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))$ for all $\mathbf{x} \in \Omega$ and $\mathbf{v} = \mathbf{0}$ on $\partial\Omega_D$, we have that

$$\delta^2E(\boldsymbol{\varphi})[\mathbf{u}] = \int_\Omega \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}^2}[\nabla\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x})] + \frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}} \cdot \nabla\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\ \mathrm{d}S(\mathbf{x}). \quad (3.24)$$

Since $\nabla\mathbf{u}(\mathbf{x}) \in \mathcal{T}(\nabla\boldsymbol{\varphi}(\mathbf{x}))$ and $\nabla\mathbf{v}(\mathbf{x}) \in \mathcal{N}(\nabla\boldsymbol{\varphi}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))$ for all $\mathbf{x} \in \Omega$, by Lemma 3.2.1 and Proposition 3.2.2 (specifically (3.19)), we have that

$$\begin{aligned}&\int_\Omega \frac{\partial^2 W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}^2}[\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})] + \frac{\partial W(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}} \cdot \nabla\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\ \mathrm{d}S(\mathbf{x})\\ &= \int_\Omega \frac{\partial^2 \widetilde{W}(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}^2}[\nabla\mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})] + \frac{\partial \widetilde{W}(\nabla\boldsymbol{\varphi}(\mathbf{x}))}{\partial\mathbf{F}} \cdot \nabla\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - \int_{\partial\Omega_T} \mathbf{t}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\ \mathrm{d}S(\mathbf{x}).\end{aligned}$$

Therefore, by (3.24),

$$\delta^2 E(\boldsymbol{\varphi})[\mathbf{u}] = \delta^2 \widetilde{E}(\boldsymbol{\varphi})[\mathbf{u}],$$

proving the first result. □

Proof of (ii). Define $\mathbf{F}_0 = \nabla\boldsymbol{\varphi}(\mathbf{x}_0)$, and let $\widetilde{W}$ be any other extension of W^{inc} (in addition to W), and let $\widetilde{\mathbb{K}}$ be given by

$$\widetilde{\mathbb{K}}[\mathbf{A}_1, \mathbf{A}_2] = \frac{\partial^2 \widetilde{W}(\mathbf{F}_0)}{\partial \mathbf{F}^2}[\mathbf{A}_1\mathbf{F}_0, \mathbf{A}_2\mathbf{F}_0] + \widetilde{p}\operatorname{tr}(\mathbf{A}_1\mathbf{A}_2),$$
$$\mathbf{A}_1, \mathbf{A}_2 \in \{\mathbf{A} \in M^{n\times n} \mid \operatorname{tr}(\mathbf{A}) = 0\},$$

where $\widetilde{p}$ is such that

$$\left(\frac{\partial \widetilde{W}(\mathbf{F}_0)}{\partial \mathbf{F}}\mathbf{F}_0^T - \widetilde{p}\mathbb{1}\right)\mathbf{N} = |\mathbf{F}_0^T\mathbf{N}|\mathbf{t}(\mathbf{x}_0). \tag{3.25}$$

Let $\mathbf{w} : G \to \mathbb{R}^n$ be solenoidal and vanish on ∂G_D. Then for p given by (1.51),

$$\begin{aligned}
\sum_{i,j,\alpha,\beta}^{n} \int_G \frac{\partial W(\mathbf{F}_0)}{\partial F_{i\alpha}} \frac{\partial}{\partial y_\beta}\left(\frac{\partial w_i}{\partial y_j} w_j\right) \nabla\boldsymbol{\varphi}(\mathbf{x}_0)_{\beta\alpha}\, \mathrm{d}\mathbf{y}& \\
= \sum_{i,j,\alpha,\beta}^{n} \int_{\partial G_T} \frac{\partial w_i}{\partial y_j} w_j \frac{\partial W(\mathbf{F}_0)}{\partial F_{i\alpha}} \nabla\boldsymbol{\varphi}(\mathbf{x}_0)_{\beta\alpha} N_\beta\, \mathrm{d}S(\mathbf{y})& \\
= \sum_{i,j}^{n} \int_{\partial G_T} \frac{\partial w_i}{\partial y_j} w_j \left(p N_i - |\mathbf{F}_0^T\mathbf{N}| t_i\right)\, \mathrm{d}S(\mathbf{y})& \\
= \sum_{i,j}^{n} \int_G p \frac{\partial}{\partial y_i}\left(\frac{\partial w_i}{\partial y_j} w_j\right)\, \mathrm{d}\mathbf{y} - \int_{\partial G_T} \frac{\partial w_i}{\partial y_j} w_j |\mathbf{F}_0^T\mathbf{N}| t_i\, \mathrm{d}S(\mathbf{y}).&
\end{aligned}$$

Hence, by (1.50),

$$\begin{aligned}
\int_G \mathbb{K}[\nabla\mathbf{w}, \nabla\mathbf{w}]\, \mathrm{d}\mathbf{y} = \int_G \frac{\partial^2 W(\mathbf{F}_0)}{\partial \mathbf{F}^2}[\nabla\mathbf{w}\mathbf{F}_0, \nabla\mathbf{w}\mathbf{F}_0] + \frac{\partial W(\mathbf{F}_0)}{\partial \mathbf{F}} \cdot (\nabla\mathbf{w}_1\mathbf{F}_0)\, \mathrm{d}\mathbf{y}& \\
+ \int_{\partial G_T} |\mathbf{F}_0^T\mathbf{N}|\mathbf{t} \cdot \mathbf{w}_1\, \mathrm{d}S(\mathbf{y})&
\end{aligned} \tag{3.26}$$

The same calculation involving $\widetilde{W}$, $\widetilde{p}$, $\widetilde{\mathbb{K}}$ gives

$$\int_G \widetilde{\mathbb{K}}[\nabla\mathbf{w}, \nabla\mathbf{w}]\ \mathrm{d}\mathbf{y} = \int_G \frac{\partial^2 \widetilde{W}(\mathbf{F}_0)}{\partial \mathbf{F}^2}[\nabla\mathbf{w}\mathbf{F}_0, \nabla\mathbf{w}\mathbf{F}_0] + \frac{\partial \widetilde{W}(\mathbf{F}_0)}{\partial \mathbf{F}} \cdot (\nabla\mathbf{w}_1\mathbf{F}_0)\ \mathrm{d}\mathbf{y}$$
$$+ \int_{\partial G_T} |\mathbf{F}_0^T\mathbf{N}|\mathbf{t} \cdot \mathbf{w}_1\ \mathrm{d}S(\mathbf{y}). \tag{3.27}$$

By Lemma 3.2.3, we have that $\nabla\mathbf{w}\mathbf{F}_0 \in \mathcal{T}(\mathbf{F}_0)$ and $\nabla\mathbf{w}_1\mathbf{F}_0 \in \mathcal{N}(\mathbf{F}_0, \nabla\mathbf{w}\mathbf{F}_0)$, where $\mathcal{T}(\mathbf{F}_0)$ and $\mathcal{N}(\mathbf{F}_0, \nabla\mathbf{w}\mathbf{F}_0)$ are given by (3.1) and (3.2), respectively. Therefore, by Proposition 3.2.2, (3.16) holds. This, together with (3.26) and (3.27), gives that

$$\int_G \mathbb{K}[\nabla\mathbf{w}, \nabla\mathbf{w}]\ \mathrm{d}\mathbf{y} = \int_G \widetilde{\mathbb{K}}[\nabla\mathbf{w}, \nabla\mathbf{w}]\ \mathrm{d}\mathbf{y}.$$

□

3.2.1 Extension invariance of incompressible, isotropic stored energy functions

For the rest of this chapter, we resume the assumption of isotropy, so there exists a symmetric function $\Phi^{\mathrm{inc}} : \Lambda_n \to \mathbb{R}$ such that (1.10) holds, where Λ_n is given by (1.9). We assume that the extension W of W^{inc} is also isotropic, so there exists a symmetric function $\Phi \in C^2((0,\infty)^n, \mathbb{R})$ such that

$$W(\mathbf{F}) = \Phi(v_1(\mathbf{F}), \ldots, v_n(\mathbf{F})), \quad \text{for all } \mathbf{F} \in M_+^{n\times n} \tag{3.28}$$

where $v_i(\mathbf{F})$, for $i = 1, \ldots, n$, are eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$. In particular, by (1.10), (3.4), and (3.28),

$$\Phi^{\mathrm{inc}}(v_1, \ldots, v_n) = \Phi(v_1, \ldots, v_n), \quad \text{for all } \mathbf{v} \in \Lambda_n, \tag{3.29}$$

where Λ_n is given by (1.9). Define

$$\mathbf{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in M_1^{n\times n}. \tag{3.30}$$

Given such $\lambda_1, \ldots, \lambda_n$ in (3.30) and the extended function Φ, we will make use of the notation (2.5). In the work to follow, we will often take another isotropic extension of W^{inc} (in addition to W), denoted by $\widetilde{W}$. This implies that there exists a symmetric function $\widetilde{\Phi} \in C^2((0,\infty)^n, \mathbb{R})$ such that

$$\widetilde{W}(\mathbf{F}) = \widetilde{\Phi}(v_1(\mathbf{F}), \ldots, v_n(\mathbf{F})), \quad \text{for all } \mathbf{F} \in M_+^{n\times n} \tag{3.31}$$

where $v_i(\mathbf{F})$, for $i = 1, \ldots, n$, are eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$. In a similar way to (2.5), we denote[3] (with no sum on repeated indices)

$$\begin{aligned} \widetilde{\Phi}_i &= \widetilde{\Phi}_{,i}(\lambda_1, \ldots, \lambda_n), & \widetilde{\Phi}_{ij} &= \widetilde{\Phi}_{,ij}(\lambda_1, \ldots, \lambda_n), & & i, j = 1, \ldots, n, \\ \widetilde{\Psi}_{ij} &= \frac{\lambda_i \widetilde{\Phi}_i - \lambda_j \widetilde{\Phi}_j}{\lambda_i^2 - \lambda_j^2}, & \widetilde{\Theta}_{ij} &= \frac{\lambda_j \widetilde{\Phi}_i - \lambda_i \widetilde{\Phi}_j}{\lambda_i^2 - \lambda_j^2}, & & i \neq j. \end{aligned} \tag{3.32}$$

It is of interest to study the relationship between the two extensions Φ and $\widetilde{\Phi}$.

Lemma 3.2.5. *Let $\epsilon > 0$, and let $\mathbf{v} : (-\epsilon, \epsilon) \to \Lambda_n$ be a one-parameter family of vectors satisfying*

$$v_1(t) \ldots v_n(t) = 1, \qquad t \in (-\epsilon, \epsilon), \tag{3.33}$$

and such that $\mathbf{v}(0) = (\lambda_1, \ldots, \lambda_n)^T$. Then

$$\dot{\mathbf{v}}(0) \in \mathcal{T}(\boldsymbol{\lambda}),$$

and

$$\ddot{\mathbf{v}}(0) \in \mathcal{N}(\boldsymbol{\lambda}, \dot{\mathbf{v}}(0)),$$

where

$$\mathcal{T}(\boldsymbol{\lambda}) = \left\{ \mathbf{u} \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_n} \end{pmatrix} \cdot \mathbf{u} = 0 \right\}, \tag{3.34}$$

$$\mathcal{N}(\boldsymbol{\lambda}, \mathbf{u}) = \left\{ \mathbf{w} \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} \frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_n} \end{pmatrix} \cdot \mathbf{w} = \begin{pmatrix} \frac{1}{\lambda_1^2} \\ \vdots \\ \frac{1}{\lambda_n^2} \end{pmatrix} \cdot \begin{pmatrix} u_1^2 \\ \vdots \\ u_n^2 \end{pmatrix} \right\}. \tag{3.35}$$

Proof. Differentiating the constraint (3.33) with respect to t once, then once more, gives

$$\begin{aligned} \sum_{i=1}^n \frac{\dot{v}_i(t)}{v_i(t)} &= 0, \\ \sum_{i=1}^n \left(\frac{\ddot{v}_i(t)}{v_i(t)} - \frac{\dot{v}_i(t)^2}{v_i(t)^2} \right) &= 0, \end{aligned}$$

respectively. Setting $t = 0$ gives the result. $\square$

[3] As in Chapter 2, (2.5), in the case when $\lambda_i = \lambda_j$ for some $i \neq j$, one interprets $\widetilde{\Psi}_{ij}$ and $\widetilde{\Theta}_{ij}$ as a limit $\lambda_j \to \lambda_i$.

Proposition 3.2.6. *Let $W : M_+^{n\times n} \to \mathbb{R}$ and $\widetilde{W} : M_+^{n\times n} \to \mathbb{R}$ be isotropic and satisfy (3.4). Let $\Phi : (0,\infty)^n \to \mathbb{R}$ satisfy (3.28), and let $\widetilde{\Phi} : (0,\infty)^n \to \mathbb{R}$ satisfy (3.31). Let $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$. Then*

$$\begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} \cdot \mathbf{u} = \begin{pmatrix} \widetilde{\Phi}_1 \\ \vdots \\ \widetilde{\Phi}_n \end{pmatrix} \cdot \mathbf{u}, \quad \textit{for all } \mathbf{u} \in \mathcal{T}(\boldsymbol{\lambda}), \tag{3.36}$$

where Φ_i, for $i = 1, \ldots, n$, are given by (2.5), $\widetilde{\Phi}_i$, for $i = 1, \ldots, n$, are given by (3.32), and $\mathcal{T}(\boldsymbol{\lambda})$ is given by (3.34).

Proof. Let $\epsilon > 0$, and let $\mathbf{v} : (-\epsilon, \epsilon) \to \Lambda_n$ be a one-parameter family of vectors satisfying (3.33), such that $\mathbf{v}(0) = (\lambda_1, \ldots, \lambda_n)^T$. Then by (3.29), for all $t \in (-\epsilon, \epsilon)$,

$$\Phi(v_1(t), \ldots, v_n(t)) = \widetilde{\Phi}(v_1(t), \ldots, v_n(t)). \tag{3.37}$$

Differentiating (3.37) with respect to t and setting $t = 0$ gives

$$\sum_{i=1}^{n} \Phi_i \dot{v}_i(0) = \sum_{i=1}^{n} \widetilde{\Phi}_i \dot{v}_i(0),$$

By Lemma 3.2.5, $\dot{\mathbf{v}}(0) \in \mathcal{T}(\boldsymbol{\lambda})$, hence we have (3.34). $\square$

Proposition 3.2.7. *Let $W : M_+^{n\times n} \to \mathbb{R}$ and $\widetilde{W} : M_+^{n\times n} \to \mathbb{R}$ be isotropic and satisfy (3.4). Let $\Phi : (0,\infty)^n \to \mathbb{R}$ satisfy (3.28), and let $\widetilde{\Phi} : (0,\infty)^n \to \mathbb{R}$ satisfy (3.31). Let $(\lambda_1, \ldots, \lambda_n) \in \Lambda_n$. Then*

$$\sum_{i,j=1}^{n} \Phi_{ij} u_i u_j + \sum_{i=1}^{n} \Phi_i u_i = \sum_{i,j=1}^{n} \widetilde{\Phi}_{ij} u_i u_j + \sum_{i=1}^{n} \widetilde{\Phi}_i w_i, \quad \textit{for all } \mathbf{u} \in \mathcal{T}(\boldsymbol{\lambda}),\ \mathbf{w} \in \mathcal{T}(\boldsymbol{\lambda}, \mathbf{u}). \tag{3.38}$$

where Φ_i and Φ_{ij}, for $i, j = 1, \ldots, n$, are given by (2.5), $\widetilde{\Phi}_i$ and $\widetilde{\Phi}_{ij}$, for $i, j = 1, \ldots, n$, are given by (3.32), $\mathcal{T}(\boldsymbol{\lambda})$ is given by (3.34), and $\mathcal{N}(\boldsymbol{\lambda}, \mathbf{u})$ is given by (3.35).

Proof. Let $\epsilon > 0$, and let $\mathbf{v} : (-\epsilon, \epsilon) \to \Lambda_n$ be a one-parameter family of vectors satisfying (3.33), and such that $\mathbf{v}(0) = (\lambda_1, \ldots, \lambda_n)^T$. By differentiating (3.37) twice and setting $t = 0$, we have that

$$\sum_{i,j=1}^{n} \Phi_{ij} \dot{v}_i(0) \dot{v}_j(0) + \sum_{i=1}^{n} \Phi_i \ddot{v}_i(0) = \sum_{i,j=1}^{n} \widetilde{\Phi}_{ij} \dot{v}_i(0) \dot{v}_j(0) + \sum_{i=1}^{n} \widetilde{\Phi}_i \ddot{v}_i(0).$$

By Lemma 3.2.5, $\dot{\mathbf{v}}(0) \in \mathcal{T}(\boldsymbol{\lambda})$, and $\ddot{\mathbf{v}}(0) \in \mathcal{N}(\boldsymbol{\lambda}, \dot{\mathbf{v}}(0))$. Hence we have (3.38). $\square$

Proposition 3.2.6 and Proposition 3.2.7 will be used to show that the result of Theorem 3.3.4 is extension invariant (see Remark 3.3.6).

3.3 Agmon's condition for incompressible, isotropic hyperelasticity

Consider an incompressible, hyperelastic, homogeneous, isotropic material in n dimensions, where $n = 2$ or 3, with stored energy function W. We consider the incompressible version of the problem given in Section 2.1; an incompressible elastic body occupies the region $\Omega = (0, L_1) \times \cdots \times (0, L_n)$ in its reference state, and we impose the slip boundary condition (2.3) or (2.4) on the side surfaces $\partial\Omega_S$ and leave the upper and lower surface $\partial\Omega_T$ free. Let $\mathbf{x}_0 \in \partial\Omega_T$ be a point on the upper surface $x_n = L_n$, so that the normal to $\partial\Omega_T$ at $\mathbf{x}_0$ is $\mathbf{n} = \mathbf{e}_n$. A minimising deformation $\boldsymbol{\varphi}$ for this system satisfies the equilibrium equations (1.18) and the following boundary conditions: the traction-free boundary condition

$$\left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x})) - p(\mathbf{x})\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x})\right)\mathbf{n} = 0, \qquad \mathbf{x} \in \partial\Omega_T, \tag{3.39}$$

the slip boundary conditions (2.3) or (2.4), and the natural boundary condition

if $\mathbf{n} = \mathbf{2}$,

$$\mathbf{e}_2 \cdot \left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\nabla\boldsymbol{\varphi}(\mathbf{x})^T\right)\mathbf{e}_1 = p(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega_S, \tag{3.40}$$

or

if $\mathbf{n} = \mathbf{3}$,

$$\begin{cases} \mathbf{e}_2 \cdot \left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\nabla\boldsymbol{\varphi}(\mathbf{x})^T\right)\mathbf{e}_1 = p(\mathbf{x}) = \mathbf{e}_3 \cdot \left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\nabla\boldsymbol{\varphi}(\mathbf{x})^T\right)\mathbf{e}_1, \\ \qquad\qquad \mathbf{x} \in \partial\Omega_S \cap \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 = 0 \text{ or } x_1 = L_1\}, \\ \mathbf{e}_1 \cdot \left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\nabla\boldsymbol{\varphi}(\mathbf{x})^T\right)\mathbf{e}_2 = p(\mathbf{x}) = \mathbf{e}_3 \cdot \left(\bar{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\nabla\boldsymbol{\varphi}(\mathbf{x})^T\right)\mathbf{e}_2, \\ \qquad\qquad \mathbf{x} \in \partial\Omega_S \cap \{\mathbf{x} \in \mathbb{R}^3 \mid x_2 = 0 \text{ or } x_2 = L_2\}, \end{cases} \tag{3.41}$$

respectively. The boundary conditions (3.40) or (3.41) are natural boundary conditions, corresponding to zero tangential stress on $\partial\Omega_S$.

Let $\mathbf{D}$ be given by (3.30), such that the stretches $\lambda_1, \ldots, \lambda_{n-1}$ are determined by the slip boundary condition (2.3) or (2.4), and λ_n is determined by the constraint $\lambda_1 \ldots \lambda_n = 1$. Let $\boldsymbol{\varphi}^h$ be a pure homogeneous deformation given by

$$\boldsymbol{\varphi}^h(\mathbf{x}) = \mathbf{D}\mathbf{x}. \tag{3.42}$$

Then $\boldsymbol{\varphi}^h$, with the constant pressure p satisfying (1.51) with traction $\mathbf{t} \equiv \mathbf{0}$, satisfies the boundary value problem given by (1.18), (3.39), (2.3) or (2.4), and (3.40) or (3.41), respectively. Hence, we are interested in additional necessary conditions for $\boldsymbol{\varphi}^h$ to be a weak local minimiser. One such condition, by Theorem 3.1.5, is the nonnegativity of J,[4] given by (1.52) (and, by Remark 1.2.37, K). A set of necessary and sufficient conditions for the nonnegativity of K can be found in Theorem 1.2.38, one of which is Agmon's condition for incompressible elasticity. Of particular interest is the tensor $\mathbb{K}$, for which we have the following lemma.

Lemma 3.3.1. *Let $W : M_+^{n\times n} \to \mathbb{R}$ be isotropic and satisfy* (3.4)*, and let $\Phi : (0,\infty)^n \to \mathbb{R}$ be given by* (3.28)*. Let $\mathbf{D}$ be given by* (3.30)*, and let $\mathbb{K}$ be given by* (1.50)*. Then*

$$\mathbb{K}[\mathbf{A},\mathbf{A}] = \sum_{i,j=1}^{n} \Phi_{ij}(\mathbf{AD})_{ii}(\mathbf{AD})_{jj} + \sum_{i\neq j} \Psi_{ij}(\mathbf{AD})_{ij}^2 + \Theta_{ij}(\mathbf{AD})_{ij}(\mathbf{AD})_{ji} + \lambda_n\Phi_n\mathrm{tr}(\mathbf{A}^2),$$

$$\textit{for all } \mathbf{A} \in M^{n\times n} \textit{ such that } \mathrm{tr}(\mathbf{A}) = 0, \qquad (3.43)$$

where Φ_n, Φ_{ij}, Ψ_{ij}, and Θ_{ij}, for $i,j = 1,\dots,n$, are given by (2.5)*.*

Proof. A simple application of Theorem 2.1.2 on (1.50) and (1.51) yields (3.43). ☐

Our goal for the remainder of this chapter is to obtain a concise algebraic condition equivalent to Agmon's condition, that depends only on the stretches $\lambda_1,\dots,\lambda_n$, the isotropic stored energy function Φ, and its derivatives, under the assumption that $\mathbb{K}$ satisfies the Legendre-Hadamard condition. We will study Agmon's condition in two dimensions for a general (incompressible) isotropic stored energy function, and find an equivalent, yet simple, algebraic condition similar to Proposition 2.1.5 (iii). We will also take a neo-Hookean example of this, which will agree with Biot's prediction [Bio63]. For the case in three dimensions, we have not been successful in obtaining a concise result for a general isotropic material, but the neo-Hookean case has proved to be simple enough to obtain a result.

3.3.1 The two dimensional case

Suppose $n = 2$, and that the extended stored energy function W is of the general isotropic form

$$W(\mathbf{F}) = \Phi(v_1(\mathbf{F}), v_2(\mathbf{F})), \quad \text{for all } \mathbf{F} \in M_+^{2\times 2}, \qquad (3.44)$$

[4]The nonnegativity of J is the incompressible analogue of quasiconvexity at the boundary of W_0, given by (1.38)

where $v_1(\mathbf{F}), v_2(\mathbf{F})$ are the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$. Before checking Agmon's condition, we have the following lemma.

Lemma 3.3.2. *Let $W : M_+^{2\times 2} \to \mathbb{R}$ be isotropic and satisfy* (3.4)*, and let $\Phi : (0,\infty)^2 \to \mathbb{R}$ be given by* (3.44)*. Let $\mathbf{D}$ be given by* (3.30)*, and let $\mathbb{K}$ be given by* (3.43)*. Then $\mathbb{K}$ satisfies the Legendre-Hadamard condition* (1.54) *if and only if*

$$\Phi_{11} \geq 0, \quad \Phi_{22} \geq 0, \quad \Psi_{12} \geq 0, \quad \textit{and } \sqrt{\Phi_{11}\Phi_{22}} + \Psi_{12} \geq |\Phi_{12} + \Theta_{12}|. \tag{3.45}$$

Remark 3.3.3. These are nonstrict versions of the inequalities appearing in Proposition 2.1.3, since the (incompressible) Legendre-Hadamard condition is a nonstrict, incompressible analogue of the strong ellipticity condition. It may be possible to extend the proof given in [KS76] that applies to compressible strong ellipticity to also include this incompressible case, but for convenience of the reader we have included a simple proof here.

Proof. By (3.43), $\mathbb{K}$ satisfies the Legendre-Hadamard condition (1.54) if and only if for any orthogonal pair $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$,

$$\begin{aligned} 0 &\leq \mathbb{K}[\mathbf{a}\otimes\mathbf{b}\mathbf{D}, \mathbf{a}\otimes\mathbf{b}\mathbf{D}] \\ &= \lambda_1^2\Phi_{11}a_1^2b_1^2 + 2\lambda_1\lambda_2\Phi_{12}a_1b_1a_2b_2 + \lambda_2^2\Phi_{22}a_2^2b_2^2 \\ &\qquad + \Psi_{12}((\lambda_2 a_1 b_2)^2 + (\lambda_1 a_2 b_1)^2) + 2\Theta_{12}(\lambda_1\lambda_2 a_1 b_1 a_2 b_2) \\ &= \mathbf{a}^T \begin{pmatrix} \lambda_1^2\Phi_{11}b_1^2 + \lambda_2^2\Psi_{12}b_2^2 & \lambda_1\lambda_2(\Phi_{12}+\Theta_{12})b_1b_2 \\ \lambda_1\lambda_2(\Phi_{12}+\Theta_{12})b_1b_2 & \lambda_1^2\Psi_{12}b_1^2 + \lambda_2^2\Phi_{22}b_2^2 \end{pmatrix} \mathbf{a}. \end{aligned}$$

That is, the matrix appearing in the last line is positive semi-definite. This holds if and only if the matrices

$$\begin{pmatrix} \Phi_{11} & 0 \\ 0 & \Psi_{12} \end{pmatrix}, \qquad \begin{pmatrix} \Psi_{12} & 0 \\ 0 & \Phi_{22} \end{pmatrix}$$

are positive semi-definite, and for all $\mathbf{b} \in \mathbb{R}^2$,

$$\begin{aligned} 0 &\leq (\lambda_1^2\Phi_{11}b_1^2 + \lambda_2^2\Psi_{12}b_2^2)(\lambda_1^2\Psi_{12}b_1^2 + \lambda_2^2\Phi_{22}b_2^2) - \lambda_1^2\lambda_2^2(\Phi_{12}+\Theta_{12})^2 b_1^2 b_2^2 \\ &= \left(b_1^2,\ b_2^2\right) \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} b_1^2 \\ b_2^2 \end{pmatrix}, \end{aligned}$$

where $\alpha = \lambda_1^4\Phi_{11}\Psi_{12}$, $\beta = \frac{1}{2}\lambda_1^2\lambda_2^2\left[\Phi_{11}\Phi_{22} + \Psi_{12}^2 - (\Phi_{12}+\Theta_{12})^2\right]$, and $\gamma = \lambda_2^4\Phi_{22}\Psi_{12}$.

Therefore, overall, $\mathbb{K}$ satisfies the Legendre-Hadamard condition if and only if

$$\Phi_{11} \geq 0, \quad \Phi_{22} \geq 0, \quad \Psi_{12} \geq 0, \tag{3.46}$$

and the polynomial

$$p(t) := \alpha t^4 + 2\beta t^2 + \gamma$$

is nonnegative for all $t \in \mathbb{R}$. In either case of proving necessity or sufficiency, we have (3.46), so $\alpha, \gamma \geq 0$. Hence, $p(t) \geq 0$ for all $t \in \mathbb{R}$ if and only if $\beta \geq -\sqrt{\alpha\gamma}$, i.e.

$$\Phi_{11}\Phi_{22} + \Psi_{12}^2 - (\Phi_{12} + \Theta_{12})^2 \geq -2\sqrt{\Phi_{11}\Phi_{22}}\Psi_{12}. \tag{3.47}$$

The last required inequality $(3.45)_4$ follows upon rearranging (3.47) and taking the square root. $\square$

We are now able to state one of the main theorems for this chapter.

Theorem 3.3.4. *Let* $W : M_+^{2\times 2} \to \mathbb{R}$ *be isotropic and satisfy* (3.4)*, and let* $\Phi : (0,\infty)^2 \to \mathbb{R}$ *be given by* (3.44)*. Let* $\mathbf{D}$ *be given by* (3.30)*, and let the tensor* $\mathbb{K}$ *be given by* (3.43)*. Suppose that* Φ *is such that* $\mathbb{K}$ *satisfies the Legendre-Hadamard condition. Then the pair* $(\mathbb{K}, \mathbf{n})$ *satisfies Agmon's condition (Theorem 1.2.38 (2)) if and only if* $\Psi_{12} = 0$*, or*

$$\lambda_1^2\Phi_{11} - 2\Phi_{12} + \frac{1}{\lambda_1^2}\Phi_{22} + \frac{1}{\lambda_1}\Phi_1 + \left(\frac{2}{\lambda_1} - \frac{1}{\lambda_1^3}\right)\Phi_2 \geq 0. \tag{3.48}$$

Proof. Let $\alpha > 0$, and $\tau_1 \in \mathbb{R}\backslash\{0\}$. We seek solutions to (1.55) of the form $(\mathbf{z}(t), q(t)) = (\mathbf{A}e^{mt}, \mu e^{mt})$, where $\mathbf{A} \in \mathbb{C}^2$, and $\mu \in \mathbb{C}$. By (3.43) and (1.53), $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{P}$ take the form

$$\mathbf{M} = \begin{pmatrix} \lambda_2^2\Psi_{12} & 0 \\ 0 & \lambda_2^2\Phi_{22} + p_2 \end{pmatrix}$$
$$\mathbf{N} = i\tau_1 \begin{pmatrix} 0 & \lambda_1\lambda_2\Theta_{12} + p_2 \\ \lambda_1\lambda_2\Phi_{12} & 0 \end{pmatrix}$$
$$\mathbf{P} = -\tau_1^2 \begin{pmatrix} \lambda_1^2\Phi_{11} + p_2 & 0 \\ 0 & \lambda_1^2\Psi_{12} \end{pmatrix},$$

where $p_2 = \lambda_2\Phi_2$. We have three cases to consider.

Nondegenerate case: Suppose $\Psi_{12} > 0$, and that $\alpha > 0$ and $\tau_1 \neq 0$ are such that

$b_\alpha^2 \neq 4ac_\alpha$, where a, b_α, and c_α are given by

$$a = \lambda_2^2\Psi_{12}, \tag{3.49a}$$

$$b_\alpha = 2(\Phi_{12}+\Theta_{12}) - \lambda_1^2\Phi_{11} - \lambda_2^2\Phi_{22} - \left(\frac{\alpha}{\tau_1}\right)^2, \tag{3.49b}$$

$$c_\alpha = \lambda_1^2\Psi_{12} + \left(\frac{\alpha}{\tau_1}\right)^2. \tag{3.49c}$$

By Lemma A.3.6, the general solution to (1.55a) and (1.55b) is given by

$$\begin{aligned} \mathbf{z}(t) &= k\begin{pmatrix} -m_+^2 \\ \mathrm{i}\tau_1 m_+ \end{pmatrix} e^{m_+ t} + l\begin{pmatrix} -m_-^2 \\ \mathrm{i}\tau_1 m_- \end{pmatrix} e^{m_- t}, \\ q(t) &= kC_1(m_+)e^{m_+ t} + lC_1(m_-)e^{m_- t}, \end{aligned} \tag{3.50}$$

where m_+ and m_- are given by

$$\frac{m_\pm^2}{\tau_1^2} = \frac{-b_\alpha \pm \sqrt{b_\alpha^2 - 4ac_\alpha}}{2a}, \tag{3.51}$$

$C_1(m)$ is given by

$$C_1(m) = \mathrm{i}\tau_1\left(\left(\Phi_{12}+\Theta_{12}-\lambda_2^2\Phi_{22}\right)m^2 + \lambda_1^2\Psi_{12}\tau_1^2 + \alpha^2\right), \tag{3.52}$$

and $k, l \in \mathbb{C}$ are arbitrary. To solve the full boundary value problem (1.55), we seek nonzero $k, l \in \mathbb{C}$ such that (1.55c) is satisfied. Substituting (3.50) into this gives us the condition that, for some nonzero pair $k, l \in \mathbb{C}$,

$$\begin{pmatrix} \lambda_2^2\Psi_{12}\left(m_+^2+\tau_1^2\right)m_+ & \lambda_2^2\Psi_{12}\left(m_-^2+\tau_1^2\right)m_- \\ \lambda_1^2\Psi_{12}\tau_1^2 + \alpha^2 + \lambda_2^2\Psi_{12}m_+^2 & \lambda_1^2\Psi_{12}\tau_1^2 + \alpha^2 + \lambda_2^2\Psi_{12}m_-^2 \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} = \mathbf{0},$$

necessitating that the above matrix is singular. Hence, a nonzero solution to (1.55c) exists if and only if for some $\alpha > 0$ and $\tau_1 \neq 0$ such that $b_\alpha^2 \neq 4ac_\alpha$,

$$\widetilde{g}\left(\left(\frac{\alpha}{\tau_1}\right)^2\right) = 0, \tag{3.53}$$

where $\widetilde{g}$ is given by

$$\widetilde{g}(x) = (c_0 + x - a)\sqrt{a(c_0+x)} + (c_0+x)(2a - b_0 + x), \tag{3.54}$$

and a, b_0, and c_0 are given by (3.49) (with $\alpha = 0$).

Degenerate case: Now suppose $\Psi_{12} > 0$, and that $\alpha > 0$, $\tau_1 \neq 0$ are such that $b_\alpha^2 = 4ac_\alpha$, where a, b_α, and c_α are given by (3.49). Then, by Lemma A.3.7, the general solution to (1.55a) and (1.55b) is given by

$$\mathbf{z}(t) = \left[k_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -m_+ \\ \mathrm{i}\tau_1 \end{pmatrix} + k_1 t \begin{pmatrix} -m_+ \\ i\tau_1 \end{pmatrix}\right] e^{m_+ t},$$
$$q(t) = \left[k_1 \left(\mathrm{i}b_\alpha \tau_1 + \frac{1}{m_+^2} C_1(m_+)\right) + \frac{k_2}{m_+} C_1(m_+) + \frac{k_1 t}{m_+} C_1(m_+)\right] e^{m_+ t},$$

where m_+ is given by (3.51), $C_1(m)$ is given by (3.52), and $k_1, k_2 \in \mathbb{C}$ are arbitrary. To solve the full boundary value problem (1.55), we seek nonzero $k_1, k_2 \in \mathbb{C}$ such that (1.55c) is satisfied. This holds if and only if k_1, k_2 satisfy

$$\begin{pmatrix} -2m_+ & -(m_+^2 + \tau_1^2) \\ \mathrm{i}\tau_1 \left(a + \frac{b_\alpha}{2}\right) & \mathrm{i}\tau_1 m_+ \left(a - \frac{b_\alpha}{2}\right) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \mathbf{0}. \tag{3.55}$$

Hence, there exists a nonzero solution to (1.55c) in the case $b_\alpha^2 = 4ac_\alpha$ if and only if the matrix appearing in (3.55) is singular. Using the fact that $\left(\frac{m_+}{\tau_1}\right)^2 = -\frac{b_\alpha}{2a}$, and $b_\alpha^2 = 4ac_\alpha$, this is if and only if

$$b_\alpha + a - 3c_\alpha = 0. \tag{3.56}$$

Note that, for this case when α and τ_1 are such that $b_\alpha^2 = 4ac_\alpha$, we have that $\widetilde{g}\left(\left(\frac{\alpha}{\tau_1}\right)^2\right) = \frac{b_\alpha}{2}\left(b_\alpha + a - 3c_\alpha\right)$. Since both $a > 0$ and $c_\alpha > 0$, we must have $b_\alpha \neq 0$. Hence, in the case $b_\alpha^2 = 4ac_\alpha$, (3.56) holds if and only if $\widetilde{g}\left(\left(\frac{\alpha}{\tau_1}\right)^2\right) = 0$.

Trivial case: Suppose $\Psi_{12} = 0$. By Lemma A.3.5, the general solution to (1.55a) and (1.55b) is given by

$$\begin{aligned} \mathbf{z}(t) &= k \begin{pmatrix} -m_0 \\ \mathrm{i}\tau_1 \end{pmatrix} e^{m_0 t}, \\ q(t) &= k \frac{C_1(m_0)}{m_0} e^{m_0 t}, \end{aligned} \tag{3.57}$$

where m_0 is given by

$$\frac{m_0}{\tau_1} = \sqrt{-\frac{c_\alpha}{b_\alpha}}, \tag{3.58}$$

$C_1(m)$ is given by (3.52), b_α is given by $(3.49)_2$, and c_α is given by $(3.49)_3$. We seek a particular solution of this form that satisfies (1.55c), so by substituting in the solution (3.57) into this and simplifying, we require a nonzero $k \in \mathbb{C}$ that

satisfies
$$\left[(\lambda_2^2\Phi_{22}+p_2)\mathrm{i}\tau_1 m_0 - \Phi_{12}\mathrm{i}\tau_1 m_0 + \frac{C_1(m_0)}{m_0}\right]k = 0.$$

Simplifying further, this condition is equivalent to requiring $\alpha^2\tau_1 = 0$, which cannot occur. Hence, *there do not exist nonzero solutions in the trivial case* $\Psi_{12} = 0$.

Overall, we have that the pair $(\mathbb{K}, \mathbf{n})$ satisfies Agmon's condition if and only if $\Psi_{12} = 0$ (the trivial case), or for all $\alpha > 0$ and $\tau_1 \neq 0$, $\widetilde{g}\left(\left(\frac{\alpha}{\tau_1}\right)^2\right) \neq 0$ (the nondegenerate and degenerate case). We have that $\widetilde{g}(x) \to +\infty$ as $x \to +\infty$, and by Lemma A.3.2, $\widetilde{g}$ is strictly increasing if $\widetilde{g}(0) \geq 0$. Therefore, the pair $(\mathbb{K}, \mathbf{n})$ satisfy Agmon's condition if and only if $\Psi_{12} = 0$ or $\widetilde{g}(0) \geq 0$, proving the theorem. □

Remark 3.3.5. Dowaikh and Ogden [DO90, equation (6.12)] also obtain (3.48) (with strict inequality) as a sufficient condition for the existence of (Rayleigh) surface waves to the "equations of incremental motion" for a general isotropic material, assuming strong ellipticity. Note that Dowaikh and Ogden [DO90] seek nonzero wave speeds to 'incremental motion', which leads to seeking nonzero solutions to (1.55) for $-\lambda_1^2\Psi_{12} \leq \alpha^2 \leq 0$, where α is now purely imaginary. Furthermore, they work in three dimensions, with the half space occupying the region $x_2 < 0$, but assume variations are independent of x_3 and have zero $\mathbf{e}_3$ component. Their conclusion is identical to (3.48) due to the fact that both problems require a nonzero solution to an eigenvalue problem (and since strong ellipticity implies $\Psi_{12} > 0$).

Remark 3.3.6. Let $\mathbf{u} = (\lambda_1, -\lambda_2)$ and $\mathbf{w} = (\lambda_1, \lambda_2)$. Then
$$\begin{pmatrix} \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \end{pmatrix} \cdot \mathbf{u} = 0,$$
so $\mathbf{u} \in \mathcal{T}(\boldsymbol{\lambda})$, and
$$\begin{pmatrix} \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \end{pmatrix} \cdot \mathbf{w} - \begin{pmatrix} \frac{1}{\lambda_1^2} \\ \frac{1}{\lambda_2^2} \end{pmatrix} \cdot \begin{pmatrix} u_1^2 \\ u_2^2 \end{pmatrix} = 0$$
so $\mathbf{w} \in \mathcal{N}(\boldsymbol{\lambda}, \mathbf{u})$. Therefore, by (3.36),
$$\lambda_1\Phi_1 - \frac{1}{\lambda_1}\Phi_2 = \sum_{i=1}^{2}\Phi_i u_i = \sum_{i=1}^{2}\widetilde{\Phi}_i u_i = \lambda_1\widetilde{\Phi}_1 - \frac{1}{\lambda_1}\widetilde{\Phi}_2. \tag{3.59}$$

and by (3.38),

$$\sum_{i,j=1}^{2} \Phi_{,ij} u_i u_j + \sum_{i=1}^{2} \Phi_i w_i = \sum_{i,j=1}^{2} \widetilde{\Phi}_{,ij} u_i u_j + \sum_{i=1}^{2} \widetilde{\Phi}_i w_i,$$

i.e.

$$\lambda_1^2 \Phi_{11} - 2\Phi_{12} + \frac{1}{\lambda_1^2}\Phi_{22} + \lambda_1 \Phi_1 + \frac{1}{\lambda_1}\Phi_2 = \lambda_1^2 \widetilde{\Phi}_{11} - 2\widetilde{\Phi}_{12} + \frac{1}{\lambda_1^2}\widetilde{\Phi}_{22} + \lambda_1 \widetilde{\Phi}_1 + \frac{1}{\lambda_1}\widetilde{\Phi}_2. \quad (3.60)$$

By multiplying (3.59) by $(\frac{1}{\lambda_1^2} - 1)$ and adding it to (3.60), we see that

$$\begin{aligned} \lambda_1^2 \Phi_{11} - 2\Phi_{12} + &\frac{1}{\lambda_1^2}\Phi_{22} + \frac{1}{\lambda_1}\Phi_1 + \left(\frac{2}{\lambda_1} - \frac{1}{\lambda_1^3}\right)\Phi_2 \\ &= \lambda_1^2 \widetilde{\Phi}_{11} - 2\widetilde{\Phi}_{12} + \frac{1}{\lambda_1^2}\widetilde{\Phi}_{22} + \frac{1}{\lambda_1}\widetilde{\Phi}_1 + \left(\frac{2}{\lambda_1} - \frac{1}{\lambda_1^3}\right)\widetilde{\Phi}_2. \end{aligned}$$

Hence, (3.48) is invariant with respect to the choice of extension.

Remark 3.3.7. A simple example of Theorem 3.3.4 can be seen by considering the incompressible neo-Hookean stored energy function

$$\Phi(\lambda_1, \lambda_2) = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right). \quad (3.61)$$

It is trivial to check that this form of Φ guarantees that $\mathbb{K}$ satisfies the Legendre-Hadamard condition, with the aid of Lemma 3.3.2. Furthermore, for Φ of the form (3.61), we have $\Psi_{12} = 1 > 0$ for any λ_1, λ_2. Hence by (3.48) with Φ of the form (3.61), Agmon's condition is satisfied if and only if

$$\frac{\lambda_2}{\lambda_1} \leq r^*,$$

where $r^* \approx 3.383$, which corresponds to when

$$\lambda_1 \geq 0.544, \quad (3.62)$$

agreeing with Biot instability, as expected (see (2.45) and (2.47)). The lack of a multiple of material constant μ in (3.61) is due to the fact that it trivially factors out of the inequalities (3.45) and (3.48); it is the same notion as in Remark 2.1.8.

Remark 3.3.8. By (1.11), we may write Φ in terms of the first principal invariant of $\mathbf{D}^T\mathbf{D}$,

$$\Phi(\lambda_1, \lambda_2) = h(I_1) = h(\lambda_1^2 + \lambda_2^2),$$

which, by (2.5), gives

$$\Phi_i = 2\lambda_i h'(I_1), \quad \Phi_{ii} = 2h'(I_1) + 4\lambda_i^2 h''(I_1), \quad \Phi_{12} = 4h''(I_1), \quad \Psi_{12} = 2h'(I_1).$$

Then (3.48) holds if and only if

$$2(\lambda_1^2 - \lambda_1^{-2})^2 h''(I_1) + (\lambda_1^2 + 1 + 3\lambda_1^{-2} - \lambda_1^{-4})h'(I_1) \geq 0. \tag{3.63}$$

Note that $\lambda_1^2 + 1 + 3\lambda_1^{-2} - \lambda_1^{-4} = -\lambda_1^2 \widetilde{F}(\lambda_1^{-2})$, where $\widetilde{F}$ is given by (2.46) and has one real root $r^* \approx 3.383$. It is immediately clear from this, that when Φ is of the form (3.61), Agmon's condition holds if and only if $\frac{\lambda_2}{\lambda_1} \leq r^*$, (again) being Biot's prediction for wrinkling instability [Bio63]. Another type of stored energy function to consider is one of the form[5]

$$h(I_1) = \frac{\mu}{2} I_1 + \frac{\nu}{4} I_1^2, \tag{3.64}$$

where $\mu, \nu \geq 0$ are material constants. This gives

$$h'(I_1) = \frac{\mu}{2} + \frac{\nu}{2} I_1, \qquad h''(I_1) = \frac{\nu}{2}.$$

Therefore, using Theorem 3.3.4, since $\Psi_{12} = \mu + \nu I_1 > 0$, Agmon's condition holds if and only if

$$\lambda_1^6 + \lambda_1^4 + 3\lambda_1^2 - 1 + \frac{\nu}{\mu}\left(3\lambda_1^8 + \lambda_1^6 + 5 + \lambda_1^{-2}\right) \geq 0. \tag{3.65}$$

We plot the region where Agmon's condition holds for various values of $\frac{\nu}{\mu}$ in Figure 3-1[6]. The limiting case where $\frac{\nu}{\mu} \to \infty$ gives that Agmon's condition holds if and only if $3\lambda_1^8 + \lambda_1^6 + 5 + \lambda_1^{-2} \geq 0$, i.e.

$$\lambda_1 \geq 0.662.$$

As $\frac{\nu}{\mu}$ increases from 0 to ∞, the critical stretch corresponding to a lower bound for λ_1 where Agmon's condition holds appears to (monotonically) increase from 0.544 to 0.662. If an incompressible 'Yeoh' stored energy function of the form (3.64) is a more accurate model than neo-Hookean for rubber elastomers used in experiments such as Gent and Cho [GC99], the disparity between wrinkling and creasing is not as large as previously thought. Figure 3-2 shows a fairly rapid growth of this lower bound as $\frac{\nu}{\mu}$ increases from 0: even a ratio of $\frac{\nu}{\mu} = \frac{1}{10}$ yields 0.601 as a lower bound for λ_1 for which Agmon's condition holds.

Remark 3.3.9. The supplementary condition (Theorem 1.2.38 (3)) is automatically true

[5] A special type of 'Yeoh' material [Yeo93].

[6] Obtained by 3D-plotting the function $z = \lambda_1^6 + \lambda_1^4 + 3\lambda_1^2 - 1 + t\left(3\lambda_1^8 + \lambda_1^6 + 5 + \lambda_1^{-2}\right)$ for $t = \frac{\nu}{\mu} \in (0, 3)$ and $\lambda_1 \in \left(\frac{1}{2}, \frac{3}{4}\right)$, and comparing with $z = 0$.

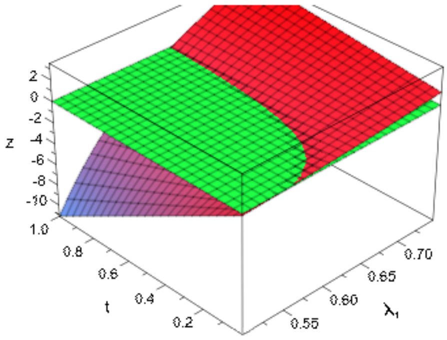


Figure 3-1: The portion of the graph in red above the green plane is the region where (3.65) holds, with $t = \frac{\nu}{\mu}$.

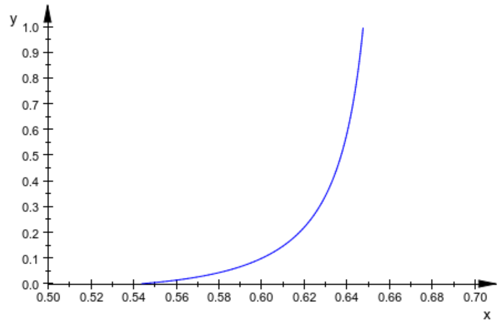


Figure 3-2: The plot of the solution to $x^6 + x^4 + 3x^2 - 1 + y\left(3x^8 + x^6 + 5 + x^{-2}\right) = 0$. Note that $y = 0$ corresponds to $x = 0.544$.

with $\pi(\boldsymbol{\tau}) \equiv 0$ for an isotropic material, since for any nonzero $\boldsymbol{\tau} \in \mathbb{R}^2$ orthogonal to $\mathbf{n}$,

$$\mathbb{K}[\boldsymbol{\tau} \otimes \mathbf{n}, \boldsymbol{\tau} \otimes \mathbf{n}] = \lambda_2^2 \Psi_{12} \tau_1^2,$$

and

$$\mathbb{K}[\boldsymbol{\tau} \otimes \mathbf{n}] = \begin{pmatrix} 0 & \lambda_2^2 \Psi_{12} \tau_1 \\ \lambda_2^2 \Psi_{12} \tau_1 & 0 \end{pmatrix}.$$

Therefore, for a general isotropic material, under the assumption of the Legendre-Hadamard condition, by Theorem 3.1.5 and Theorem 1.2.38, $\boldsymbol{\varphi}^h$ is a weak local minimiser of the functional E only if $\Psi_{12} = 0$, or if (3.48) holds.

3.3.2 The three dimensional case

For an incompressible, isotropic stored energy function W^{inc}, and its isotropic extension W in three dimensions, we have that there exists a symmetric function Φ satisfying

$$W(\mathbf{F}) = \Phi(v_1(\mathbf{F}), v_2(\mathbf{F}), v_3(\mathbf{F})), \quad \text{for all } \mathbf{F} \in M_+^{3\times 3}, \tag{3.66}$$

where $v_i(\mathbf{F})$, for $i = 1, 2, 3$, are the eigenvalues of $\sqrt{\mathbf{F}^T\mathbf{F}}$. Denote $p_3 = \lambda_3 \Phi_3$. We have by (1.53) that $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{P}$ are given by

$$\mathbf{M} = \begin{pmatrix} \lambda_3^2\Psi_{13} & 0 & 0 \\ 0 & \lambda_3^2\Psi_{23} & 0 \\ 0 & 0 & \lambda_3^2\Phi_{33} + p_3 \end{pmatrix}, \tag{3.67}$$

$$\mathbf{N} = i \begin{pmatrix} 0 & 0 & (\lambda_1\lambda_3\Theta_{13} + p_3)\tau_1 \\ 0 & 0 & (\lambda_2\lambda_3\Theta_{23} + p_3)\tau_2 \\ \lambda_1\lambda_3\Phi_{13}\tau_1 & \lambda_2\lambda_3\Phi_{23}\tau_2 & 0 \end{pmatrix}, \tag{3.68}$$

$$\mathbf{P} = -\begin{pmatrix} (\lambda_1^2\Phi_{11} + p_3)\tau_1^2 + \lambda_2^2\Psi_{12}\tau_2^2 & (\lambda_1\lambda_2(\Phi_{12} + \Theta_{12}) + p_3)\tau_1\tau_2 & 0 \\ (\lambda_1\lambda_2(\Phi_{12} + \Theta_{12}) + p_3)\tau_1\tau_2 & \lambda_1^2\Psi_{12}\tau_1^2 + (\lambda_2^2\Phi_{22} + p_3)\tau_2^2 & 0 \\ 0 & 0 & \lambda_1^2\Psi_{13}\tau_1^2 + \lambda_2^2\Psi_{23}\tau_2^2 \end{pmatrix}. \tag{3.69}$$

Finding a concise algebraic inequality equivalent to Agmon's condition, in terms of only the stretches λ_i, Φ, and its derivatives, suffers the same problem as the three-dimensional compressible case in that it is too complex in general. Hence, in this subsection, we will consider an example isotropic stored energy function Φ of the 'neo-Hookean' form

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2). \tag{3.70}$$

We have again omitted a material constant μ in (3.70) due to the fact that it trivially factors out in the following calculations (see Remark 3.3.7).

Lemma 3.3.10. *Let Φ be of the form* (3.70)*. Let $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in M_1^{3\times 3}$, and let $\mathbb{K}$ be given by* (3.43)*. Then $\mathbb{K}$ satisfies the Legendre-Hadamard condition* (1.54)*.*

Proof. Note that for Φ of the form (3.70), by (2.5),

$$\Phi_i = \lambda_i, \quad \Phi_{ij} = \delta_{ij}, \quad \Psi_{ij} = 1, \quad \Theta_{ij} = 0. \tag{3.71}$$

Hence, by (3.43) with Φ of the form (3.70), we have that for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ such that $\mathbf{a} \cdot \mathbf{b} = 0$,

$$\begin{aligned} \mathbb{K}[\mathbf{a}\otimes\mathbf{b}, \mathbf{a}\otimes\mathbf{b}] &= \sum_{i,j=1}^{n} \delta_{ij}(\mathbf{a}\otimes\mathbf{bD})_{ii}(\mathbf{a}\otimes\mathbf{bD})_{jj} + \sum_{i\neq j}(\mathbf{a}\otimes\mathbf{bD})_{ij}^2 + \lambda_n\Phi_n\mathrm{tr}((\mathbf{a}\otimes\mathbf{b})^2) \\ &= \mathbf{a}^T \begin{pmatrix} \lambda_1^2b_1^2 + \lambda_2^2b_2^2 + \lambda_3^2b_3^2 & 0 & 0 \\ 0 & \lambda_1^2b_1^2 + \lambda_2^2b_2^2 + \lambda_3^2b_3^2 & 0 \\ 0 & 0 & \lambda_1^2b_1^2 + \lambda_2^2b_2^2 + \lambda_3^2b_3^2 \end{pmatrix} \mathbf{a}, \end{aligned}$$

which is clearly nonnegative for all orthogonal $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. □

Theorem 3.3.11. *Let Φ be of the form* (3.70)*. Let* $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in M_1^{3\times 3}$*,* $\mathbf{n} = \mathbf{e}_3$*, and let $\mathbb{K}$ be given by* (3.43)*. Then the pair* $(\mathbb{K}, \mathbf{n})$ *satisfies Agmon's condition if and only if*

$$\frac{\min\{\lambda_1, \lambda_2\}}{\lambda_3} \geq \xi^*, \tag{3.72}$$

where $\xi^* \approx 0.296$ *is the only real root of the cubic function*

$$F(s) := s^3 + s^2 + 3s - 1, \qquad s > 0. \tag{3.73}$$

Remark 3.3.12. Note that (3.72) holds if and only if (2.56) holds.

Proof. For Φ of the form (3.70), we have $p_3 = \lambda_3^2$, and by (3.67), (3.68), and (3.69),

$$\mathbf{M} = \begin{pmatrix} \lambda_3^2 & 0 & 0 \\ 0 & \lambda_3^2 & 0 \\ 0 & 0 & \lambda_3^2 + p_3 \end{pmatrix}, \tag{3.74}$$

$$\mathbf{N} = \mathrm{i} \begin{pmatrix} 0 & 0 & p_3\tau_1 \\ 0 & 0 & p_3\tau_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.75}$$

$$\mathbf{P} = -\begin{pmatrix} (\lambda_1^2 + p_3)\tau_1^2 + \lambda_2^2\tau_2^2 & p_3\tau_1\tau_2 & 0 \\ p_3\tau_1\tau_2 & \lambda_1^2\tau_1^2 + (\lambda_2^2 + p_3)\tau_2^2 & 0 \\ 0 & 0 & \lambda_1^2\tau_1^2 + \lambda_2^2\tau_2^2 \end{pmatrix}. \tag{3.76}$$

Note that the neo-Hookean stored energy function (3.70) satisfies the Legendre-Hadamard condition by Lemma 3.3.10. The method to follow is similar to that of the two-dimensional case: Let $\alpha > 0$ and $\boldsymbol{\tau} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ be orthogonal to $\mathbf{n}$. We seek a nonzero solution of the boundary value problem (1.55) that decays to zero as $t \to -\infty$.

Nondegenerate case Suppose α and $\boldsymbol{\tau}$ are such that $\sigma(\boldsymbol{\tau}, \alpha) \neq |\boldsymbol{\tau}|$, where $\sigma(\boldsymbol{\tau}, \alpha)$ is given by

$$\sigma(\boldsymbol{\tau}, \alpha) = \frac{1}{\lambda_3}\sqrt{\lambda_1^2\tau_1^2 + \lambda_2^2\tau_2^2 + \alpha^2}. \tag{3.77}$$

Then by Lemma A.3.10, the general solution of (1.55a) and (1.55b) that decays to zero as $t \to -\infty$ is given by

$$(\mathbf{z}, q) = (\mathbf{A}e^{|\boldsymbol{\tau}|t} + (\mathbf{B} + \mathbf{C})e^{\sigma(\boldsymbol{\tau},\alpha)t}, \mu e^{|\boldsymbol{\tau}|t} + (\nu + \rho)e^{\sigma(\boldsymbol{\tau},\alpha)t}),$$

where

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \mu \end{pmatrix} = k_1 \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ |\boldsymbol{\tau}| \\ \lambda_3^2(\sigma(\boldsymbol{\tau},\alpha)^2 - |\boldsymbol{\tau}|^2) \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \nu \end{pmatrix} = k_2 \begin{pmatrix} \mathrm{i}\tau_2\sigma(\boldsymbol{\tau},\alpha) \\ \mathrm{i}\tau_1\sigma(\boldsymbol{\tau},\alpha) \\ 2\tau_1\tau_2 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \rho \end{pmatrix} = k_3 \begin{pmatrix} -\mathrm{i}\sigma(\boldsymbol{\tau},\alpha)\tau_1 \\ \mathrm{i}\sigma(\boldsymbol{\tau},\alpha)\tau_2 \\ \tau_2^2 - \tau_1^2 \\ 0 \end{pmatrix},$$

and $k_1, k_2, k_3 \in \mathbb{C}$ are arbitrary. We must also solve (1.55c), so we seek nonzero $k_1, k_2, k_3 \in \mathbb{C}$ such that

$$(|\boldsymbol{\tau}|\mathbf{M} + \mathbf{N})\mathbf{A} + (\sigma(\boldsymbol{\tau},\alpha)\mathbf{M} + \mathbf{N})(\mathbf{B} + \mathbf{C}) + (\mu + \nu + \rho)\mathbf{n} = \mathbf{0}.$$

This is equivalent to requiring

$$\begin{pmatrix} 2i\tau_1|\boldsymbol{\tau}| & \mathrm{i}\tau_2(\sigma(\boldsymbol{\tau},\alpha)^2 + 2\tau_1^2) & \mathrm{i}\tau_1(\tau_2^2 - \tau_1^2 - \sigma(\boldsymbol{\tau},\alpha)^2) \\ 2i\tau_2|\boldsymbol{\tau}| & \mathrm{i}\tau_1(\sigma(\boldsymbol{\tau},\alpha)^2 + 2\tau_2^2) & \mathrm{i}\tau_2(\tau_2^2 - \tau_1^2 + \sigma(\boldsymbol{\tau},\alpha)^2) \\ |\boldsymbol{\tau}|^2 + \sigma(\boldsymbol{\tau},\alpha)^2 & 4\sigma(\boldsymbol{\tau},\alpha)\tau_1\tau_2 & 2\sigma(\boldsymbol{\tau},\alpha)(\tau_2^2 - \tau_1^2) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0.$$

We have that, for $\boldsymbol{\tau}$ and α such that $\sigma(\boldsymbol{\tau},\alpha) \neq |\boldsymbol{\tau}|$, nonzero solutions to (1.55) do not exist if and only if the determinant of the above matrix is nonzero. After simplifying, this is if and only if

$$F\left(\frac{\sigma(\boldsymbol{\tau},\alpha)}{|\boldsymbol{\tau}|}\right) \neq 0, \tag{3.78}$$

where F is given by (3.73). Since F has exactly one positive real root ξ^*, we have that (3.78) holds if and only if

$$\frac{\sigma(\boldsymbol{\tau},\alpha)}{|\boldsymbol{\tau}|} \neq \xi^*. \tag{3.79}$$

Note that, by (3.77), for any $\boldsymbol{\tau} \neq \mathbf{0}$, $\frac{\sigma(\boldsymbol{\tau},\alpha)}{|\boldsymbol{\tau}|}$ is increasing in α. Hence, (3.79) holds (when $\sigma(\boldsymbol{\tau},\alpha) \neq |\boldsymbol{\tau}|$) if and only if

$$\frac{\sigma(\boldsymbol{\tau},0)}{|\boldsymbol{\tau}|} \geq \xi^*.$$

Furthermore, we note that

$$\min_{\boldsymbol{\tau}\in\mathbb{R}^2\setminus\{\mathbf{0}\}} \left\{\frac{\sigma(\boldsymbol{\tau},0)}{|\boldsymbol{\tau}|}\right\} = \frac{\min\{\lambda_1,\lambda_2\}}{\lambda_3}.$$

Hence, for all $\boldsymbol{\tau}$, α such that $\sigma(\boldsymbol{\tau},\alpha) \neq |\boldsymbol{\tau}|$, there do not exist nonzero solutions to (1.55) if and only if (3.72) holds.

Degenerate case Now suppose $\sigma(\boldsymbol{\tau},\alpha) = |\boldsymbol{\tau}|$, where $\sigma(\boldsymbol{\tau},\alpha)$ is given by (3.77). Then by Lemma A.3.11, the general solution to (1.55a) and (1.55b) that decays to zero as $t \to -\infty$ is given by

$$(\mathbf{z},q) = ((\mathbf{U}+t\mathbf{V})e^{|\boldsymbol{\tau}|t}, (\mu+t\nu)e^{|\boldsymbol{\tau}|t}),$$

where

$$\begin{pmatrix}\mathbf{U}\\ u\end{pmatrix} = k_1\begin{pmatrix}|\boldsymbol{\tau}|\\ 0\\ -\mathrm{i}\tau_1\\ 0\end{pmatrix} + k_2\begin{pmatrix}0\\ 0\\ -1\\ -2\lambda_3^2|\boldsymbol{\tau}|\end{pmatrix} + k_3\begin{pmatrix}0\\ |\boldsymbol{\tau}|\\ -\mathrm{i}\tau_2\\ 0\end{pmatrix},$$

$$\begin{pmatrix}\mathbf{V}\\ v\end{pmatrix} = k_2\begin{pmatrix}\mathrm{i}\tau_1\\ \mathrm{i}\tau_2\\ |\boldsymbol{\tau}|\\ 0\end{pmatrix},$$

and $k_1,k_2,k_3 \in \mathbb{C}$ are arbitrary. Finally, we seek a particular nonzero solution to (1.55c). This requires at least one nonzero $k_1,k_2,k_3 \in \mathbb{C}$ such that

$$\begin{pmatrix}\lambda_3^2|\boldsymbol{\tau}|^2+p\tau_1 & 0 & p\tau_1\tau_2\\ p\tau_1\tau_2 & 0 & \lambda_3^2|\boldsymbol{\tau}|^2+p\tau_2^2\\ -i(\lambda_3^2+p)\tau_1|\boldsymbol{\tau}| & -2\lambda_3^2|\boldsymbol{\tau}| & -i(\lambda_3^2+p)\tau_2|\boldsymbol{\tau}|\end{pmatrix}\begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix} = \mathbf{0}.$$

However, the determinant of the above matrix equates to $4\lambda_3^6|\boldsymbol{\tau}|^5$, which is nonzero. Hence, nonzero solutions to (1.55) *do not exist* when $\alpha,\boldsymbol{\tau}$ are such that $\sigma(\boldsymbol{\tau},\alpha) = |\boldsymbol{\tau}|$.

We have that nonzero solutions to (1.55) which decay to zero as $t \to -\infty$ exist if and only if the complement of (3.72) holds (from the nondegenerate case; no nontrivial solutions exist in the degenerate case). Hence overall, the pair $(\mathbb{K}, \mathbf{n})$ satisfies Agmon's condition if and only if (3.72) holds.

$\square$

Remark 3.3.13. Note that $\lambda_1, \lambda_2, \lambda_3$ satisfy (3.72) if and only if (2.56) holds, by using the fact that $\lambda_1\lambda_2\lambda_3 = 1$ and $\frac{1}{\xi^*} = r^*$. Figure 2-4, therefore, is an accurate illustration for the region of (λ_1, λ_2) where Agmon's condition for an incompressible neo-Hookean material fails. We repeat the notable fact that setting λ_1 or λ_2 to 1 obtains Biot's result, which is from a two-dimensional setting.

Remark 3.3.14. By Lemma 3.3.10, when Φ is of the form (3.70), $\mathbb{K}$ satisfies the Legendre-Hadamard condition (Theorem 1.2.38, (1)). Furthermore, the supplementary condition (Theorem 1.2.38, (3)) is vacuously true, since for any nonzero $\boldsymbol{\tau} \in \mathbb{R}^3$ orthogonal to $\mathbf{n}$,

$$\mathbb{K}[\boldsymbol{\tau} \otimes \mathbf{n}, \boldsymbol{\tau} \otimes \mathbf{n}] = \lambda_3^2(\tau_1^2 + \tau_2^2) \neq 0.$$

Therefore, for a material in three dimensions with Φ of the 'neo-Hookean' form (3.70) by Theorem 3.1.5 and Theorem 1.2.38, $\boldsymbol{\varphi}^h$ is a weak local minimiser of the functional E given by (1.41) with zero traction $\mathbf{t}$ only if (3.72) holds.

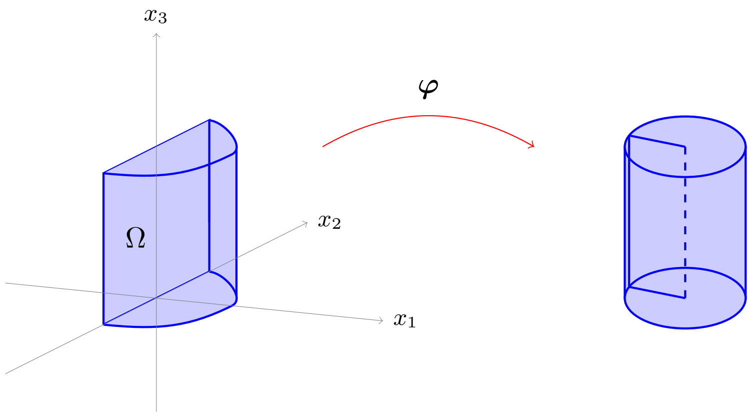
Chapter 4

The closing of a deformed half-cylinder

In this chapter, we consider an elastic body which corresponds to a half-cylinder in its reference configuration Ω. With the naturally oriented choice of cylindrical polar coordinates (r, θ, z), we consider deformations of Ω of the form

$$\boldsymbol{\varphi} = \begin{pmatrix} R(r)\cos(\Theta(\theta)) \\ R(r)\sin(\Theta(\theta)) \\ Z(z) \end{pmatrix}. \tag{4.1}$$

In Section 4.1, we consider the compressible case, and in Section 4.2, the incompressible case.



We will consider two possible boundary conditions on its flat side:

1. Ciarlet's self-contact boundary condition (see [Cia88, Section 5.6]), in which whenever two boundary points experience self-contact, the stresses acting on the

surfaces corresponding to those points balance out;

2. a forced self-contact condition, in which the two halves are fixed together pointwise.

In section 4.1, we take a compressible isotropic material, and consider one of two possible boundary conditions on its flat side; Dirichlet data and Ciarlet's self-contact boundary condition [Cia88, Section 5.6]. We also subject the (flat) ends of the half-cylinder to a slip boundary condition. A necessary condition for a deformation to be a weak local minimiser of E over deformations satisfying these boundary conditions is that it is a 'weak equilibrium solution' (see Definition 4.1.6). The main result of Section 4.1 is Theorem 4.1.7, which gives necessary and sufficient conditions for a deformation of the form (4.1) to be a weak equilibrium solution, under the assumption that the tension-extension inequalities hold (see (4.12)). In Section 4.2, we consider an incompressible isotropic material with stored energy function $h^{\mathrm{inc}}(I_1, I_2)$ assumed to be convex and monotone increasing in each argument. We suppose this material occupies a half-cylinder, subject to a slip boundary condition on its ends, and a 'forced self-contact' boundary condition on its flat side (see (4.38) or (4.67)). In the first instance, we will take the two-dimensional analogue of this problem on a half-disk, with the constraint $\det(\nabla\boldsymbol{\varphi}) = \alpha^2$ instead of incompressibility, allowing a more general extension to three dimensions later. We will prove that the double-covering[1] deformation

$$\boldsymbol{\varphi}_{DC_\alpha} = \frac{\alpha r}{\sqrt{2}} \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$$

is a global minimiser of this problem, with uniqueness up to rotation and translation. Reconsidering the problem in three dimensions, we will use the two dimensional result to prove that the deformation

$$\widetilde{\boldsymbol{\varphi}}_{DC\gamma} = \begin{pmatrix} \frac{r}{\sqrt{2\gamma}} \cos(2\theta) \\ \frac{r}{\sqrt{2\gamma}} \sin(2\theta) \\ \gamma z \end{pmatrix}$$

is a global minimiser of this problem over the set of deformations of the form

$$\boldsymbol{\varphi}(\mathbf{x}) = \begin{pmatrix} \varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \\ \zeta(x_3) \end{pmatrix},$$

[1]Named as such by Bevan [Bev14], since it maps a half-disk to a half-disk twice over.

with uniqueness up to rotation and translation.

Throughout this chapter, for $t \in \mathbb{R}$, we shall denote by $\mathbf{Q}(t)$ the real matrix in $SO(2)$, given by

$$\mathbf{Q}(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix},$$

and in $SO(3)$, given by

$$\mathbf{Q}(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

appropriately used for two and three dimensional cases, respectively. The following two lemmas supplement calculating the gradient of a deformation in cylindrical coordinates.

Lemma 4.0.1. *Let $\boldsymbol{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ be a map. Let $\mathbf{x} \in \mathbb{R}^2$ be a point given in cartesian coordinates, and let $\boldsymbol{\varphi}(\mathbf{x})$ denote cartesian coordinates of the mapped point. Let $\mathbf{r} = (r, \theta)$ denote polar coordinates corresponding to $\mathbf{x}$. Define $\boldsymbol{\psi}(\mathbf{r}) = \boldsymbol{\psi}(r, \theta) = (R(r, \theta), \Theta(r, \theta))$ by*

$$\begin{pmatrix} \varphi_1(\mathbf{x}) \\ \varphi_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} R(r,\theta)\cos(\Theta(r,\theta)) \\ R(r,\theta)\sin(\Theta(r,\theta)) \end{pmatrix},$$

i.e. the deformed polar coordinates corresponding to $\boldsymbol{\varphi}(\mathbf{x})$. Then

$$\nabla \boldsymbol{\varphi} = \mathbf{Q}(\Theta)\,\mathrm{diag}(1, R) \nabla_{\mathbf{r}} \boldsymbol{\psi}(\mathbf{r})\,\mathrm{diag}\left(1, \tfrac{1}{r}\right) \mathbf{Q}(\theta)^T, \tag{4.2}$$

where $\nabla_{\mathbf{r}} \boldsymbol{\psi}$ is the deformation gradient with respect to both deformed and undeformed polar coordinates

$$\nabla_{\mathbf{r}} \boldsymbol{\psi}(\mathbf{r}) = \begin{pmatrix} \frac{\partial R}{\partial r} & \frac{\partial R}{\partial \theta} \\ \frac{\partial \Theta}{\partial r} & \frac{\partial \Theta}{\partial \theta} \end{pmatrix}.$$

Proof. Note that

$$\left(\frac{\partial x_i}{\partial r_j}\right) = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = \mathbf{Q}(\theta)\,\mathrm{diag}(1, r), \tag{4.3}$$

and similarly,

$$\left(\frac{\partial \varphi_i}{\partial \psi_j}\right) = \begin{pmatrix} \cos(\Theta) & -R\sin(\Theta) \\ \sin(\Theta) & R\cos(\Theta) \end{pmatrix} = \mathbf{Q}(\Theta)\,\mathrm{diag}(1, R), \tag{4.4}$$

Hence, by virtue of the chain rule, (4.3), and (4.4),

$$\begin{aligned}\nabla\boldsymbol{\varphi} &= \left(\frac{\partial\varphi_i}{\partial\psi_j}\right)\left(\frac{\partial\psi_j}{\partial r_\beta}\right)\left(\frac{\partial r_\beta}{\partial x_\alpha}\right)\\ &= \mathbf{Q}(\Theta)\operatorname{diag}(1,R)\nabla_{\mathbf{r}}\boldsymbol{\psi}(\mathbf{r})\left(\mathbf{Q}(\theta)\operatorname{diag}(1,r)\right)^{-1}\\ &= \mathbf{Q}(\Theta)\operatorname{diag}(1,R)\nabla_{\mathbf{r}}\boldsymbol{\psi}(\mathbf{r})\operatorname{diag}\left(1,\tfrac{1}{r}\right)\mathbf{Q}(\theta)^T.\end{aligned}$$

as required. □

Lemma 4.0.2. *Let $\boldsymbol{\varphi}:\mathbb{R}^3\to\mathbb{R}^3$ be a map. Let $\mathbf{x}\in\mathbb{R}^3$ be a point given in cartesian coordinates, and let $\boldsymbol{\varphi}(\mathbf{x})$ denote cartesian coordinates of the mapped point. Let $\mathbf{r}=(r,\theta,z)$ denote cylindrical polar coordinates corresponding to $\mathbf{x}$. Define $\boldsymbol{\psi}(\mathbf{r})=\boldsymbol{\psi}(r,\theta,z)=(R(r,\theta,z),\Theta(r,\theta,z),Z(r,\theta,z))$ by*

$$\begin{pmatrix}\varphi_1(\mathbf{x})\\ \varphi_2(\mathbf{x})\\ \varphi_3(\mathbf{x})\end{pmatrix}=\begin{pmatrix}R(r,\theta,z)\cos(\Theta(r,\theta,z))\\ R(r,\theta,z)\sin(\Theta(r,\theta,z))\\ Z(r,\theta,z)\end{pmatrix},$$

i.e. the deformed cylindrical polar coordinates corresponding to $\boldsymbol{\varphi}(\mathbf{x})$. Then

$$\nabla\boldsymbol{\varphi}=\mathbf{Q}(\Theta)\operatorname{diag}(1,R,1)\nabla_{\mathbf{r}}\boldsymbol{\psi}(\mathbf{r})\operatorname{diag}\left(1,\tfrac{1}{r},1\right)\mathbf{Q}(\theta)^T, \tag{4.5}$$

where $\nabla_{\mathbf{r}}\boldsymbol{\psi}$ is the deformation gradient with respect to both deformed and undeformed cylindrical polar coordinates

$$\nabla_{\mathbf{r}}\boldsymbol{\psi}(\mathbf{r})=\begin{pmatrix}\frac{\partial R}{\partial r} & \frac{\partial R}{\partial\theta} & \frac{\partial R}{\partial z}\\ \frac{\partial\Theta}{\partial r} & \frac{\partial\Theta}{\partial\theta} & \frac{\partial\Theta}{\partial z}\\ \frac{\partial Z}{\partial r} & \frac{\partial Z}{\partial\theta} & \frac{\partial Z}{\partial z}\end{pmatrix}.$$

Proof. The proof is a trivial extension of the same steps followed in the proof of Lemma 4.0.1. □

4.1 Compressible double-covering maps

4.1.1 Cylindrical deformations satisfying the equilibrium equations

First, we seek necessary conditions for a deformation of the form

$$\boldsymbol{\varphi}=\begin{pmatrix}R(r)\cos(\Theta(\theta))\\ R(r)\sin(\Theta(\theta))\\ Z(z)\end{pmatrix} \tag{4.6}$$

to be a solution of the compressible equilibrium equations, which, in terms of deformed cylindrical coordinates, are given by

$$\frac{\partial}{\partial R}T_{11}^{(R)} + \frac{1}{R}\frac{\partial}{\partial \Theta}T_{12}^{(R)} + \frac{\partial}{\partial Z}T_{13}^{(R)} + \tfrac{1}{R}\left(T_{11}^{(R)} - T_{22}^{(R)}\right) = 0, \tag{4.7a}$$

$$\frac{\partial}{\partial R}T_{12}^{(R)} + \frac{1}{R}\frac{\partial}{\partial \Theta}T_{22}^{(R)} + \frac{\partial}{\partial Z}T_{23}^{(R)} + \frac{2}{R}T_{12}^{(R)} = 0, \tag{4.7b}$$

$$\frac{\partial}{\partial R}T_{13}^{(R)} + \frac{1}{R}\frac{\partial}{\partial \Theta}T_{23}^{(R)} + \frac{\partial}{\partial Z}T_{33}^{(R)} + \frac{1}{R}T_{13}^{(R)} = 0, \tag{4.7c}$$

where $\mathbf{T}^{(R)}$ is the radial Cauchy stress tensor, given by $\mathbf{T}^{(R)} = \mathbf{Q}(\Theta)^T\mathbf{T}\mathbf{Q}(\Theta)$. For a proof of the equivalence of (1.17) and (4.7), see Lemma A.4.1, which can be found in the appendix. Furthermore, if $\boldsymbol{\varphi}$ is of the form (4.6), note that by (4.5),

$$\begin{aligned}\widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}) &= \widehat{\mathbf{S}}\left(\mathbf{Q}(\Theta)\operatorname{diag}\left(R'(r), \tfrac{R(r)\Theta'(\theta)}{r}, Z'(z)\right)\mathbf{Q}(\theta)^T\right)\\ &= \mathbf{Q}(\Theta)\operatorname{diag}\left(g_1(r,\theta,z), g_2(r,\theta,z), g_3(r,\theta,z)\right)\mathbf{Q}(\theta)^T,\end{aligned} \tag{4.8}$$

where $g_i(r,\theta,z)$ is defined by

$$g_i(r,\theta,z) = \Phi_{,i}\left(R'(r), \frac{R(r)\Theta'(\theta)}{r}, Z'(z)\right), \qquad i = 1,2,3. \tag{4.9}$$

Therefore,

$$\begin{aligned}\mathbf{T}^{(R)} &= \mathbf{Q}(\Theta)\frac{1}{\det(\nabla\boldsymbol{\varphi}(\mathbf{x}))}\widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))(\nabla\boldsymbol{\varphi}(\mathbf{x}))^T\mathbf{Q}(\Theta)^T\\ &= \operatorname{diag}\left(\tfrac{r}{R\Theta'Z'}g_1(r,\theta,z), \tfrac{1}{R'Z'}g_2(r,\theta,z), \tfrac{r}{RR'\Theta'}g_3(r,\theta,z)\right).\end{aligned} \tag{4.10}$$

Lemma 4.1.1. *A deformation $\boldsymbol{\varphi}$ of the form* (4.6) *is a solution of the equilibrium equations if and only if the following three equations are satisfied:*

$$\frac{\partial}{\partial r}\left(rg_1(r,\theta,z)\right) - \Theta'(\theta)g_2(r,\theta,z) = 0, \tag{4.11a}$$

$$\frac{\partial}{\partial \theta}g_2(r,\theta,z) = 0, \tag{4.11b}$$

$$\frac{\partial}{\partial z}g_3(r,\theta,z) = 0, \tag{4.11c}$$

where g_i, for $i = 1,2,3$ are given by (4.9).

Proof. Using (4.10), a deformation of the form (4.6) satisfies the equilibrium equations

(4.7) if and only if

$$
\begin{aligned}
0 &= \frac{\partial}{\partial R} T_{11}^{(R)} + \frac{1}{R}\left(T_{11}^{(R)} - T_{22}^{(R)}\right) \\
&= \frac{1}{RR'\Theta' Z'}\left(\frac{\partial}{\partial r}(r g_1(r,\theta,z)) - \Theta' g_2(r,\theta,z)\right), \\
0 &= \frac{1}{R}\frac{\partial}{\partial \Theta} T_{22}^{(R)} \\
&= \frac{1}{RR'\Theta' Z'}\frac{\partial}{\partial \theta} g_2(r,\theta,z) \\
0 &= \frac{\partial}{\partial Z} T_{33}^{(R)} \\
&= \frac{r}{RR'\Theta' Z'}\frac{\partial}{\partial z} g_3(r,\theta,z).
\end{aligned}
$$

Since $\det(\nabla \boldsymbol{\varphi}) = \frac{RR'\Theta' Z'}{r} > 0$, this yields the result. □

Proposition 4.1.2. *Let Φ satisfy the tension-extension inequalities (with no sum on repeated indices)*

$$
\Phi_{,ii}(v_1, v_2, v_3) > 0, \qquad \text{for } v_1, v_2, v_3 \in (0,\infty),\ i = 1,2,3. \tag{4.12}
$$

Suppose a deformation $\boldsymbol{\varphi}$ of the form (4.6) *satisfies the equilibrium equations. Then $\Theta(\theta) = \beta\theta + \alpha$ and $Z(z) = \gamma z + z_0$, where α, β, γ, and z_0 are real constants.*

Proof. By Lemma 4.1.1, we have that (R, Θ, Z) satisfy (4.11). Expanding (4.11b), we have that

$$
0 = \frac{\partial}{\partial \theta}\Phi_{,2}\left(R'(r), \tfrac{R(r)\Theta'(\theta)}{r}, Z'(z)\right) = \frac{R(r)\Theta''(\theta)}{r}\Phi_{,22}\left(R'(r), \tfrac{R(r)\Theta'(\theta)}{r}, Z'(z)\right),
$$

and similarly with (4.11c),

$$
0 = \frac{\partial}{\partial z}\Phi_{,3}\left(R'(r), \tfrac{R(r)\Theta'(\theta)}{r}, Z'(z)\right) = Z''(z)\Phi_{,33}\left(R'(r), \tfrac{R(r)\Theta'(\theta)}{r}, Z'(z)\right).
$$

By the tension-extension inequalities, $\Phi_{,22}, \Phi_{,33} > 0$. So we must have that $\Theta'' \equiv 0$ and $Z'' \equiv 0$, hence the result. □

Remark 4.1.3. If indeed the tension-extension inequalities hold, and if $\boldsymbol{\varphi}$ is a solution of the equilibrium equations of the form (4.6), Proposition 4.1.2 implies that (4.9) is in fact a function of r only,

$$
g_i(r) = \Phi_{,i}\left(R'(r), \frac{\beta R(r)}{r}, \gamma\right),
$$

and so we omit the dependence on θ and z from now on. In this case, the equations (4.11b) and (4.11c) become trivially satisfied, and (4.11a) becomes an equation in r only. We summarise this in the following corollary.

Corollary 4.1.4. *Suppose Φ satisfies the tension-extension inequalities* (4.12)*, and the deformation $\boldsymbol{\varphi}$ is of the form* (4.6)*. Then $\boldsymbol{\varphi}$ satisfies the equilibrium equations if and only if*

$$\boldsymbol{\varphi} = \begin{pmatrix} R(r)\cos(\beta\theta+\alpha) \\ R(r)\sin(\beta\theta+\alpha) \\ \gamma z + z_0 \end{pmatrix}, \tag{4.13}$$

where $R(r), \beta$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r g_1(r)\right) = \beta g_2(r). \tag{4.14}$$

4.1.2 Cylindrical weak equilibria

For this subsection, we will consider deformations on the half-cylinder

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1,\ x_1 > 0,\ 0 < x_3 < L\}. \tag{4.15}$$

For brevity, we denote by Γ the half-disk

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1,\ x_1 > 0\}, \tag{4.16}$$

so that $\Omega = \Gamma \times (0, L)$. We will partition the boundary of Ω as

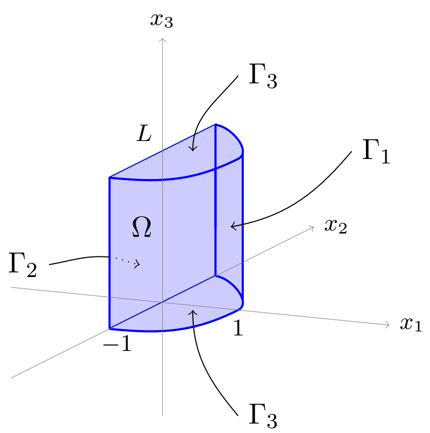


Figure 4-1: The half-cylinder Ω.

$$\Gamma_1 = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1,\ x_1 > 0,\ 0 < x_3 < L\},$$
$$\Gamma_2 = \{\mathbf{x} \in \mathbb{R}^3 \mid 0 < x_2 < 1,\ x_1 = 0,\ 0 < x_3 < L\} = \{0\} \times (-1, 1) \times (0, L),$$
$$\Gamma_3 = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1,\ x_1 > 0,\ x_3 = 0 \text{ or } x_3 = L\} = \Gamma \times \{0, L\}.$$

That is, Γ_1 is the curved surface with normal $\mathbf{e}_r$, Γ_2 is the flat surface with normal $-\mathbf{e}_1$, and Γ_3 is the top and bottom (half-disk) ends, with normals $\mathbf{e}_3$ and $-\mathbf{e}_3$, respectively. See figure (4-1).

We will consider deformations on Ω that satisfy the following set of boundary conditions. We subject Γ_3 to a slip boundary condition

$$\varphi_3(x_1, x_2, 0) = 0,\ \varphi_3(x_1, x_2, L) = \gamma_0 L, \qquad \text{for } (x_1, x_2) \in \Gamma, \tag{4.17}$$

for some given $\gamma_0 > 0$, and Γ_1 is left free. We will consider two separate cases of boundary conditions on Γ_2:

Case 1: Dirichlet data:

$$\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\varphi}_0(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Gamma_2, \tag{4.18}$$

where $\boldsymbol{\varphi}_0$ is some given map;

Case 2: Self-contact:

$$\begin{gathered} \text{For all } \mathbf{x}, \mathbf{y} \in \Gamma_2 \text{ such that } \boldsymbol{\varphi}^{-1}(\boldsymbol{\varphi}(\mathbf{x})) = \{\mathbf{x}, \mathbf{y}\}, \\ \widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\mathbf{n}(\mathbf{x}) + \widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{y}))\mathbf{n}(\mathbf{y}) = \mathbf{0}. \end{gathered} \tag{4.19}$$

The condition (4.19) is Ciarlet's "self-contact boundary condition" (see [Cia88, Section 5.6]),[2] which is when Γ_2 deforms such that two separate surface elements meet, and guarantees that the forces exerted on each surface element balance. For the rest of this section, we will refer to each case as "case 1" and "case 2" respectively.

We define the set of admissible deformations for either case by

Case 1:

$$\begin{aligned} \mathcal{A}_1 = \{\boldsymbol{\varphi} \in C^2(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3) \mid\ & \boldsymbol{\varphi} \text{ is of the form (4.6)}, \\ & \boldsymbol{\varphi} \text{ satisfies (4.17) and (4.18)}\}, \end{aligned} \tag{4.20}$$

[2]Ciarlet [Cia88, Section 5.6] also shows that the stress acting on each element is parallel to the normal of the deformed surface.

Case 2:

$$\begin{aligned}\mathcal{A}_2 = \{\boldsymbol{\varphi} \in C^2(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3) \mid \boldsymbol{\varphi} &\text{ is of the form (4.6)},\\ \boldsymbol{\varphi} &\text{ satisfies (4.17) and (4.19)}\}.\end{aligned} \tag{4.21}$$

Remark 4.1.5. To define an appropriate set of variations, suppose that

$$\int_{\Gamma_2} \frac{\partial W(\nabla \boldsymbol{\varphi}(\mathbf{x}))}{\partial \mathbf{F}} \cdot (\mathbf{u}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x}))\, \mathrm{d}S(\mathbf{x}) = 0 \tag{4.22}$$

for all $\mathbf{u} \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$ such that

$$\begin{gathered}\text{for all } \mathbf{x}, \mathbf{y} \in \Gamma_2 \text{ such that } \boldsymbol{\varphi}^{-1}(\boldsymbol{\varphi}(\mathbf{x})) = \{\mathbf{x}, \mathbf{y}\},\\ \mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{y}).\end{gathered} \tag{4.23}$$

Define $\mathcal{L} \subseteq \Gamma_2$ to be the set of $\mathbf{x} \in \Gamma_2$ such that there is exactly one point $\mathbf{y} \neq \mathbf{x}$ such that $\boldsymbol{\varphi}(\mathbf{y}) = \boldsymbol{\varphi}(\mathbf{x})$. Then there exist sets L_-, L_+ such that $L_- \cup L_+ = \mathcal{L}$, $L_- \cap L_+ = \emptyset$, and $\boldsymbol{\varphi}(L_-) = \boldsymbol{\varphi}(L_+)$. Then we may write the integral in (4.22) as

$$\int_{\mathcal{L} \cup (\Gamma_2 \setminus \mathcal{L})} \frac{\partial W(\nabla \boldsymbol{\varphi}(\mathbf{x}))}{\partial \mathbf{F}} \cdot (\mathbf{u}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x}))\, \mathrm{d}S(\mathbf{x}).$$

Choose $\mathbf{u}$ compactly supported on $\mathcal{L}$, so the integral over $\Gamma_2 \setminus \mathcal{L}$ may be neglected. Hence by (4.22),

$$\int_{L_-} \widehat{\mathbf{S}}(\nabla \boldsymbol{\varphi}(\mathbf{x})) \cdot (\mathbf{u}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x}))\, \mathrm{d}S(\mathbf{x}) + \int_{L_+} \widehat{\mathbf{S}}(\nabla \boldsymbol{\varphi}(\mathbf{y})) \cdot (\mathbf{u}(\mathbf{y}) \otimes \mathbf{n}(\mathbf{y}))\, \mathrm{d}S(\mathbf{y}) = 0. \tag{4.24}$$

Note that if $\mathbf{x} \in L_-$, $\mathbf{y} \in L_+$, and $\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{y})$, then $\mathbf{n}(\mathbf{x}) = -\mathbf{n}(\mathbf{y})$. Therefore, by considering $\mathbf{u}$ nonzero only on arbitrarily small neighbourhoods of any point $\mathbf{x} \in L_-$, with corresponding $\mathbf{y} \in L_+$ such that $\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{y})$, we have that (4.24) implies

$$\widehat{\mathbf{S}}(\nabla \boldsymbol{\varphi}(\mathbf{x}))\mathbf{n}(\mathbf{x}) + \widehat{\mathbf{S}}(\nabla \boldsymbol{\varphi}(\mathbf{y}))\mathbf{n}(\mathbf{y}) = \mathbf{0},$$

whenever $\boldsymbol{\varphi}^{-1}(\boldsymbol{\varphi}(x)) = \{\mathbf{x}, \mathbf{y}\}$, $\mathbf{x}, \mathbf{y} \in \partial\Omega$. That is, $\boldsymbol{\varphi}$ satisfies (4.19).

Remark 4.1.5 allows us to classify variations in the case when a potential minimiser gives rise to self-contact. Considering our system before on the half-cylinder Ω given by (4.15), we have the following definition.

Definition 4.1.6. The deformation $\boldsymbol{\varphi} \in \mathcal{A}_i$ is a *weak equilibrium solution* if $\frac{\partial W(\nabla \boldsymbol{\varphi})}{\partial F_{i\alpha}} \in L^1(\Omega)$ for $i, \alpha = 1, 2, 3$, and

$$\int_{\Omega} \frac{\partial W(\nabla \boldsymbol{\varphi}(\mathbf{x}))}{\partial \mathbf{F}} \cdot \nabla \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0$$

for all $\mathbf{u} \in \mathcal{V}_i$, $i = 1$ or 2, where

Case 1:

$$\mathcal{V}_1 = \{\mathbf{u} \in C^2(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3) \mid \mathbf{u} = 0 \text{ on } \Gamma_2, \; u_3 = 0 \text{ on } \Gamma_3\},$$

Case 2:

$$\mathcal{V}_2 = \{\mathbf{u} \in C^2(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3) \mid \mathbf{u} \text{ satisfies (4.23)}, \; u_3 = 0 \text{ on } \Gamma_3\}.$$

We are now ready to state the main theorem of this section.

Theorem 4.1.7. *Suppose* Φ *satisfies the tension-extension inequalities* (4.12)*, and the deformation* $\boldsymbol{\varphi} \in \mathcal{A}_1$ *or* $\mathcal{A}_2$*. Then* $\boldsymbol{\varphi}$ *is a weak equilibrium solution if and only if* $\boldsymbol{\varphi}$ *is of the form* (4.13) *with* $\gamma = \gamma_0$*,* $z_0 = 0$*,* $rg_1(r) \in W^{1,1}(0,1)$*,* $g_2 \in L^1(0,1)$*,* R, β *satisfy the boundary value problem*

$$\frac{\mathrm{d}}{\mathrm{d}r}(rg_1(r)) = \beta g_2(r), \qquad r \in (0,1), \tag{4.25a}$$

$$g_1(1) = 0, \tag{4.25b}$$

with $0 < \beta \leq 2$*, and*

Case 1: *(*$\boldsymbol{\varphi}_0 \in \mathcal{A}_1$*)* $\boldsymbol{\varphi}$ *is also of the form* (4.13)*, with its corresponding* R_0, β_0 *satisfying* (4.25a)*;*

Case 2: *(*$\boldsymbol{\varphi} \in \mathcal{A}_2$*)* $\lim_{\epsilon \to 0} \epsilon g_1(\epsilon) = 0$*, and if* $\beta < 2$*, then* $g_2(r) = 0$ *for* $r \in (0,1)$*.*

Remark 4.1.8. In both cases, R, β must satisfy (4.25a) and a natural boundary condition of zero stress on Γ_1, given by (4.25b). Case 1 corresponds to Dirichlet data on the flat surface Γ_2 at any given intermediate angle β, and the radial displacement boundary condition must satisfy (4.25a). Case 2 corresponds to when the flat surface Γ_2 is left free, requiring a zero stress boundary condition except for the case $\beta = 2$, where Γ_2 'closes up' and results in self-contact. Indeed, when $\beta = 2$, the self-contact condition on Γ_2 is automatically satisfied. The following proof is similar to [Bal82, Theorem 4.2].

Proof of Theorem 4.1.7: Necessity. Note that there exist $\mathbf{x}, \mathbf{y} \in \Gamma_2$ such that

$\boldsymbol{\varphi}^{-1}(\boldsymbol{\varphi}(\mathbf{x})) = \{\mathbf{x}, \mathbf{y}\}$ if and only if there exists $m \in \mathbb{Z}$ such that

$$\Theta\left(\frac{\pi}{2}\right) = \Theta\left(-\frac{\pi}{2}\right) + 2\pi m. \tag{4.26}$$

Let $\mathbf{u} \in \mathcal{V}_1$ or $\mathcal{V}_2$ (respective to each case), and define $\boldsymbol{\psi}$ by

$$\boldsymbol{\psi}(r, \theta, z) = \mathbf{u}(x_1, x_2, x_3), \quad \mathbf{x} \in \Omega, \tag{4.27}$$

where r, θ, z are polar coordinates in the reference configuration. Then $\boldsymbol{\psi} \in C^\infty(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3)$ is such that $\psi_3(r, \theta, 0) = 0 = \psi_3(r, \theta, L)$ and

Case 1 $\mathbf{u} \in \mathcal{V}_1 \implies \boldsymbol{\psi}(r, \pm\frac{\pi}{2}, z) = 0$;

Case 2 $\mathbf{u} \in \mathcal{V}_2 \implies \boldsymbol{\psi}(r, -\frac{\pi}{2}, z) = \boldsymbol{\psi}(r, \frac{\pi}{2}, z)$ if (4.26) holds.

We have

$$\nabla \mathbf{u} = \frac{\partial \boldsymbol{\psi}}{\partial r} \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial \boldsymbol{\psi}}{\partial \theta} \otimes \mathbf{e}_\theta + \frac{\partial \boldsymbol{\psi}}{\partial z} \otimes \mathbf{e}_z,$$

so by (4.8),

$$\frac{\partial W(\nabla \boldsymbol{\varphi})}{\partial \mathbf{F}} \cdot \nabla \mathbf{u} = g_1(r, \theta, z)\frac{\partial \boldsymbol{\psi}}{\partial r} \cdot \mathbf{e}_R + \frac{g_2(r, \theta, z)}{r}\frac{\partial \boldsymbol{\psi}}{\partial \theta} \cdot \mathbf{e}_\Theta + g_3(r, \theta, z)\frac{\partial \psi_3}{\partial z}, \tag{4.28}$$

where $\mathbf{e}_R = (\cos(\Theta(\theta)), \sin(\Theta(\theta)), 0)^T$, $\mathbf{e}_\Theta = (-\sin(\Theta(\theta)), \cos(\Theta(\theta)), 0)^T$.

We will make three different choices of $\boldsymbol{\psi}$, parallel to the principal directions $\mathbf{e}_R$, $\mathbf{e}_\Theta$, and $\mathbf{e}_z$, to obtain the result.

1. Let $\boldsymbol{\psi} = h(r)\tau(\theta)l(z)\mathbf{e}_R$, where $l \in C^\infty(0, L) \cap C([0, L])$,

   Case 1: $h \in C^\infty(0, 1) \cap C([0, 1])$ with $h(0) = 0$, and $\tau \in C_0^\infty(-\frac{\pi}{2}, \frac{\pi}{2})$

   Case 2: $h \in C^\infty(0, 1) \cap C([0, 1])$, and $\tau \in C^\infty(-\frac{\pi}{2}, \frac{\pi}{2}) \cap C([-\frac{\pi}{2}, \frac{\pi}{2}])$, such that $\tau(-\frac{\pi}{2}) = \tau(\frac{\pi}{2})$ if (4.26) holds.

   Note that

   $$\frac{\partial \boldsymbol{\psi}}{\partial r} = h'(r)\tau(\theta)l(z)\mathbf{e}_R, \tag{4.29}$$
   $$\frac{\partial \boldsymbol{\psi}}{\partial \theta} = h(r)\tau'(\theta)l(z)\mathbf{e}_R + \Theta'(\theta)h(r)\tau(\theta)l(z)\mathbf{e}_\Theta, \tag{4.30}$$

   and $\psi_3 = 0$. Since $\boldsymbol{\varphi}$ is a weak equilibrium solution, we have by (4.28), (4.29), and (4.30) that

$$
\begin{aligned}
0 &= \int_\Omega g_1(r,\theta,z)\frac{\partial \boldsymbol{\psi}}{\partial r}\cdot \mathbf{e}_R + \frac{g_2(r,\theta,z)}{r}\frac{\partial \boldsymbol{\psi}}{\partial \theta}\cdot \mathbf{e}_\Theta + g_3(r,\theta,z)\frac{\partial \psi_3}{\partial z}\ \mathrm{d}\mathbf{x} \\
&= \int_0^L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r g_1(r,\theta,z)h'(r)\tau(\theta)l(z) + g_2(r,\theta,z)\Theta'(\theta)h(r)\tau(\theta)l(z) r\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= \int_0^L l(z) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tau(\theta) \int_0^1 r g_1(r,\theta,z)h'(r) + \Theta'(\theta) g_2(r,\theta,z)h(r)\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}z.
\end{aligned}
$$

By arbitrariness of h, τ, and l, and by integrating the first term by parts with respect to r, in both cases we deduce that

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(r g_1(r,\theta,z)\right) &= \Theta'(\theta) g_2(r,\theta,z), \\
g_1(1,\theta,z) &= 0,
\end{aligned}
\tag{4.31}
$$

the prior being exactly (4.11a). In case 2 only, we also deduce that for any $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ and $z \in (0,L)$, $\lim_{\epsilon\to 0} \epsilon g_1(\epsilon,\theta,z) = 0$. Furthermore, since $\frac{\partial W(\nabla \boldsymbol{\varphi})}{\partial F_{i\alpha}} \in L^1(\Omega)$ for $i,\alpha = 1,2,3$, we have that for any $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ and $z \in (0,L)$, $r g_1(r,\theta,z), g_2(r,\theta,z) \in L^1(0,1)$.

2. We now choose $\boldsymbol{\psi} = h(r)\tau(\theta)l(z)\mathbf{e}_\Theta$, with the same h, τ, and l as the previous step. This time,

$$
\frac{\partial \boldsymbol{\psi}}{\partial r} = h'(r)\tau(\theta)l(z)\mathbf{e}_\Theta, \tag{4.32}
$$

$$
\frac{\partial \boldsymbol{\psi}}{\partial \theta} = h(r)\tau'(\theta)l(z)\mathbf{e}_\Theta - \Theta'(\theta)h(r)\tau(\theta)l(z)\mathbf{e}_R. \tag{4.33}
$$

Therefore, since $\boldsymbol{\varphi}$ is a weak equilibrium solution, we have by (4.28), (4.32), and (4.33) that

$$
\begin{aligned}
0 &= \int_\Omega g_1(r,\theta,z)\frac{\partial \boldsymbol{\psi}}{\partial r}\cdot \mathbf{e}_R + \frac{g_2(r,\theta,z)}{r}\frac{\partial \boldsymbol{\psi}}{\partial \theta}\cdot \mathbf{e}_\Theta + g_3(r,\theta,z)\frac{\partial \psi_3}{\partial z}\ \mathrm{d}\mathbf{x} \\
&= \int_0^L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \left(\frac{g_2(r,\theta,z)}{r} h(r)\tau'(\theta)l(z)\right) r\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= \int_0^L l(z)\int_0^1 h(r) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tau'(\theta) g_2(r,\theta,z)\ \mathrm{d}\theta\mathrm{d}r\mathrm{d}z,
\end{aligned}
$$

so by arbitrariness of h, τ, and l, in both cases we obtain (4.11b). In case 2 only, we also deduce that

$$g_2(r, \tfrac{\pi}{2}, z) = g_2(r, -\tfrac{\pi}{2}, z), \qquad \text{if (4.26) holds,} \tag{$\dagger$}$$
$$g_2(r, \tfrac{\pi}{2}, z) = g_2(r, -\tfrac{\pi}{2}, z) = 0, \qquad \text{otherwise.} \tag{$\ddagger$}$$

3. Finally, we choose $\boldsymbol{\psi} = h(r)\tau(\theta)l(z)\mathbf{e}_z$, with the same h and τ as before, but now we let $l \in C_0^\infty(0, L)$ in order to satisfy $u_3 = 0$ on Γ_3. Then

$$\begin{aligned} 0 &= \int_\Omega g_1(r,\theta,z)\frac{\partial \boldsymbol{\psi}}{\partial r}\cdot \mathbf{e}_R + \frac{g_2(r,\theta,z)}{r}\frac{\partial \boldsymbol{\psi}}{\partial \theta}\cdot \mathbf{e}_\Theta + g_3(r,\theta,z)\frac{\partial \psi_3}{\partial z}\,\mathrm{d}\mathbf{x} \\ &= \int_0^L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r g_3(r,\theta,z)h(r)\tau(\theta)l'(z)\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z, \end{aligned}$$

leading us to deduce (4.11c) in both cases.

In summary, in both cases we have (4.11), and for any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $z \in (0, L)$, $rg_1(r,\theta,z), g_2(r,\theta,z) \in L^1(0,1)$. In case 2 only, we also have that $\lim_{\epsilon\to 0} \epsilon g_1(\epsilon,\theta,z) = 0$, and ($\dagger$) or ($\ddagger$). Therefore, in both cases, by Proposition 4.1.2, we have that for some α, β, γ, and z_0,

$$\boldsymbol{\varphi} = \begin{pmatrix} R(r)\cos(\beta\theta + \alpha) \\ R(r)\sin(\beta\theta+\alpha) \\ \gamma z + z_0 \end{pmatrix}.$$

By Corollary 4.1.4, $R(r), \beta$ satisfy (4.25a), where g_i for $i = 1, 2, 3$ are now functions of r only. In particular, (4.31) reduces to (4.25b). Furthermore,

Case 1: $\boldsymbol{\varphi} \in \mathcal{A}_1$, so $z_0 = 0$, $\gamma = \gamma_0$. In order for the solution to be continuous, the Dirichlet boundary data (4.18) must be compatible with the inner solution, so $\boldsymbol{\varphi}$ is also of the form (4.13), with its corresponding R_0, β_0 satisfying (4.25).

Case 2: $\boldsymbol{\varphi} \in \mathcal{A}_2$, so $z_0 = 0$, $\gamma = \gamma_0$. Since $\boldsymbol{\varphi}$ is of the form (4.13), the condition for self-contact (4.26) is equivalent to when $\beta = 2$. The 'natural boundary condition' ($\dagger$) holds automatically if indeed $\beta = 2$, but if $\beta < 2$, since each g_i for $i = 1, 2, 3$ are independent of θ and z, ($\ddagger$) reduces to the condition

$$g_2(r) = 0, \quad \text{for all } r \in (0, 1),$$

as required.

□

Proof of Theorem 4.1.7: Sufficiency. By the hypothesis of the theorem, $rg_1(r), g_2 \in L^1(0,1)$. Hence by (4.8), $\frac{\partial W(\nabla\boldsymbol{\varphi})}{\partial F_{i\alpha}} \in L^1(\Omega)$ for $i, \alpha = 1, 2, 3$. Let $\mathbf{u} \in \mathcal{V}_1$ or $\mathcal{V}_2$ (respective to each case), and define $\boldsymbol{\psi}$ by (4.27). Then we have (4.28), and since $\boldsymbol{\psi} \in \mathcal{V}_1$ or $\mathcal{V}_2$, $\int_0^L \frac{\partial \psi_3}{\partial z}\,\mathrm{d}z = 0$. Therefore, noting that $\mathbf{e}_R = (\cos(\beta\theta+\alpha), \sin(\beta\theta+\alpha), 0)^T$ and $\mathbf{e}_\Theta = (-\sin(\beta\theta+\alpha), \cos(\beta\theta+\alpha), 0)^T$, we have

$$\begin{aligned}
&\int_\Omega \frac{\partial W(\nabla\boldsymbol{\varphi})}{\partial \mathbf{F}} \cdot \nabla \mathbf{u}\,\mathrm{d}\mathbf{x} = \int_\Omega g_1(r)\frac{\partial \boldsymbol{\psi}}{\partial r}\cdot \mathbf{e}_R + \frac{g_2(r)}{r}\frac{\partial \boldsymbol{\psi}}{\partial \theta}\cdot \mathbf{e}_\Theta + g_3(r)\frac{\partial \psi_3}{\partial z}\,\mathrm{d}\mathbf{x} \\
&= \lim_{\epsilon\to 0}\left[\int_0^L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_\epsilon^1 \frac{\partial}{\partial r}\left(r g_1(r)\boldsymbol{\psi}\cdot \mathbf{e}_R\right) - \frac{\mathrm{d}}{\mathrm{d}r}(r g_1(r))\boldsymbol{\psi}\cdot\mathbf{e}_R + g_2(r)\frac{\partial \boldsymbol{\psi}}{\partial \theta}\cdot \mathbf{e}_\Theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\right] \\
&= \lim_{\epsilon\to 0}\left[\int_0^L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_\epsilon^1 g_2(r)\left(\frac{\partial \boldsymbol{\psi}}{\partial \theta}\cdot \mathbf{e}_\Theta - \beta\boldsymbol{\psi}\cdot\mathbf{e}_R\right)\mathrm{d}r\right) - \epsilon g_1(\epsilon)\boldsymbol{\psi}(\epsilon,\theta,z)\cdot\mathbf{e}_R\,\mathrm{d}\theta\mathrm{d}z\right],
\end{aligned} \tag{4.34}$$

where we have used (4.28) in the first line, the chain rule to obtain the second line, and (4.25) to obtain the third line. Note that

Case 1: since $\boldsymbol{\psi} \in \mathcal{V}_1$, $\boldsymbol{\psi} = 0$ on Γ_2,

Case 2: since $\boldsymbol{\psi} \in \mathcal{V}_2$ (and by the hypothesis of the theorem),

$$\begin{cases} g_2(r) = 0 \text{ for all } r \in (0,1), & \text{if } \beta < 2, \\ \boldsymbol{\psi}(r, -\frac{\pi}{2}, z) = \boldsymbol{\psi}(r, \frac{\pi}{2}, z) \text{ for all } r \in (0,1) \text{ and } z \in (0,L), & \text{if } \beta = 2. \end{cases}$$

So in either case, for any $\epsilon \in (0,1)$,

$$\begin{aligned}
0 &= \int_0^L \int_\epsilon^1 g_2(r)\left[(\boldsymbol{\psi}\cdot\mathbf{e}_\Theta)\Big|_{\theta=\frac{\pi}{2}} - (\boldsymbol{\psi}\cdot\mathbf{e}_\Theta)\Big|_{\theta=-\frac{\pi}{2}}\right]\mathrm{d}r\mathrm{d}z \\
&= \int_0^L \int_\epsilon^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_2(r)\frac{\partial}{\partial \theta}\left(\boldsymbol{\psi}\cdot\mathbf{e}_\Theta\right)\mathrm{d}\theta\mathrm{d}r\mathrm{d}z \\
&= \int_0^L \int_\epsilon^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_2(r)\left(\frac{\partial \boldsymbol{\psi}}{\partial \theta}\cdot\mathbf{e}_\Theta - \beta\boldsymbol{\psi}\cdot\mathbf{e}_R\right)\mathrm{d}\theta\mathrm{d}r\mathrm{d}z.
\end{aligned} \tag{4.35}$$

Therefore, by (4.34) and (4.35), all that remains to prove the theorem is to show that

$$\lim_{\epsilon\to 0}\left[\int_0^L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \epsilon g_1(\epsilon)\boldsymbol{\psi}(\epsilon,\theta,z)\cdot\mathbf{e}_R\,\mathrm{d}\theta\mathrm{d}z\right] = 0. \tag{4.36}$$

Case 1: Since $\boldsymbol{\psi} \in \mathcal{V}_1$, $\boldsymbol{\psi}(0,\theta,z) = 0$. Since $rg(r) \in W^{1,1}(0,1)$, $\lim\limits_{\epsilon\to 0}\left[\epsilon g_1(\epsilon)\boldsymbol{\psi}(\epsilon,\theta,z)\cdot\mathbf{e}_R\right] = 0$. Therefore, by Sobolev embedding of $W^{1,1}(0,1)$ into the Hölder space $C^{0,\alpha}(0,1)$, (4.36) holds.

Case 2: $\lim_{\epsilon\to 0} \epsilon g_1(\epsilon) = 0$ by the hypothesis of the theorem, and $\boldsymbol{\psi} \in \mathcal{V}_2$ implies $\boldsymbol{\psi}(0,\theta,z)$ is finite for all θ, z. Hence, $\lim_{\epsilon\to 0} [\epsilon g_1(\epsilon)\boldsymbol{\psi}(\epsilon,\theta,z)\cdot \mathbf{e}_R] = 0$, and the convergence is uniform, so (4.36) holds.

$\square$

Remark 4.1.9. We note that in case 2, where Γ_2 is left stress-free, the strict case $\beta < 2$ has an additional necessary requirement that $g_2(r) = 0$. This seems to suggest that $\beta = 2$ is a special case where a (compressible) 'double-covering' deformation automatically satisfies the natural boundary condition on the free surface Γ_2.

Remark 4.1.10. An obvious extension to Theorem 4.1.7 is to allow admissible deformations of the general form

$$\boldsymbol{\varphi} = \begin{pmatrix} R(r,\theta,z)\cos(\Theta(r,\theta,z)) \\ R(r,\theta,z)\sin(\Theta(r,\theta,z)) \\ Z(r,\theta,z) \end{pmatrix}.$$

It may be possible to show that a minimiser of E over these more general deformations must be of the form (4.6), but a proof of this is currently not known.

4.2 Energy minimising properties of an incompressible half-cylinder subject to forced self-contact

In this section, we will consider incompressible deformations of a half-cylinder subject to a 'forced self-contact' boundary condition on the flat surface. Note that the self-contact boundary condition considered in the previous section requires that the stresses acting on the surfaces at two points experiencing self-contact with each other balance out, whereas the 'forced self-contact' boundary condition which we consider in this section requires that one half of the flat surface is fixed (pointwise) to the other half, but the displacement for either side is not given explicitly (see (4.38)). Following a method similar to that of Sivaloganathan and Spector [SS08b], we will prove that, given these preliminaries, the double-covering map is the global minimiser of the stored energy

$$E^{\mathrm{inc}}[\boldsymbol{\varphi}] = \int_{\Omega} W^{\mathrm{inc}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x}, \tag{4.37}$$

for a large class of stored energy functions W^{inc}.

4.2.1 The two dimensional case

We will first study the problem in two dimensions, by considering deformations $\boldsymbol{\varphi}$ on the domain Γ defined by (4.16), with boundary condition

$$\boldsymbol{\varphi}(0, x_2) = \boldsymbol{\varphi}(0, -x_2), \qquad 0 < x_2 < 1. \tag{4.38}$$

This is the forced self-contact boundary condition. For some fixed $\alpha > 0$, we suppose deformations on Γ are subjected to the constraint

$$\det(\nabla\boldsymbol{\varphi}(\mathbf{x})) = \alpha^2, \qquad \mathbf{x} \in \Gamma. \tag{4.39}$$

Note that we allow $\alpha > 0$ to be general, which includes the incompressible case $\alpha = 1$. This free parameter will be of use when considering the three dimensional problem in a later subsection.

To eliminate trivial nonuniqueness through rotation and translation, we will also impose the constraints

$$\int_\Gamma \varphi_1(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_\Gamma \varphi_2(\mathbf{x})\ \mathrm{d}\mathbf{x} = 0, \tag{4.40a}$$

$$\int_\Gamma \frac{\partial\varphi_1(\mathbf{x})}{\partial y}\ \mathrm{d}\mathbf{x} = 0 < \int_\Gamma \frac{\partial\varphi_2(\mathbf{x})}{\partial y}\ \mathrm{d}\mathbf{x}. \tag{4.40b}$$

Define the set of admissible deformations on Γ by

$$\mathcal{A}_\alpha = \{\boldsymbol{\varphi} \in C^1(\Gamma, \mathbb{R}^2) \cap C(\overline{\Gamma}, \mathbb{R}^2) \mid \boldsymbol{\varphi} \text{ satisfies (4.38), (4.39), and (4.40)}\}. \tag{4.41}$$

Define the two-dimensional *double-covering map* $\boldsymbol{\varphi}_{DC_\alpha}$ by

$$\boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x}) = \frac{\alpha}{\sqrt{2}|\mathbf{x}|}\begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix}, \qquad \mathbf{x} \in \Gamma, \tag{4.42}$$

which can be written in polar coordinates as

$$\boldsymbol{\varphi}_{DC_\alpha} = \frac{\alpha r}{\sqrt{2}}\begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}, \qquad (r,\theta) \in (0,1) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Note that $\boldsymbol{\varphi}_{DC_\alpha} \in \mathcal{A}_\alpha$, but $\boldsymbol{\varphi}_{DC_\alpha} \notin C^1(\overline{\Gamma}, \mathbb{R}^2)$. Our goal for this subsection is to prove that, for some class of stored energy functions W^{inc}, the global minimiser of the incompressible stored energy (in two dimensions)

$$E^{\mathrm{inc}}[\boldsymbol{\varphi}] = \int_{\Gamma} W^{\mathrm{inc}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}\mathbf{x} \tag{4.43}$$

over the set $\mathcal{A}_1$ is the double-covering map $\boldsymbol{\varphi}_{DC_1}$.

For $r \in (0,1)$, the half-disk with centre $(0,0)$, radius r, occupying the right-hand side of the plane is denoted by

$$\Gamma_r = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < r,\ x_1 > 0\},$$

and its curved, semi-circular boundary by

$$\mathrm{SC}_r = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = r,\ x_1 > 0\}.$$

We write $\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{a}} = (\nabla\boldsymbol{\varphi})\mathbf{a}$ for the directional derivative of $\boldsymbol{\varphi}$ in the direction $\mathbf{a}$. For any $\mathbf{x} \in \mathrm{SC}_r$, we write $\mathbf{n} = \mathbf{e}_r$ (the outward unit normal vector at $\mathbf{x}$), and $\mathbf{t} = \mathbf{e}_\theta$ (the unit tangent vector at $\mathbf{x}$ such that $(\mathbf{n}, \mathbf{t})$ is positively oriented).

Lemma 4.2.1. *Let $\boldsymbol{\varphi} \in \mathcal{A}_\alpha$, and let $r \in (0,1)$. Then for all $\mathbf{x} \in \mathrm{SC}_r$,*

$$|\nabla\boldsymbol{\varphi}(\mathbf{x})|^2 \geq \left|\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{t}}(\mathbf{x})\right|^2 + \frac{\alpha^4}{\left|\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{t}}(\mathbf{x})\right|^2}, \tag{4.44}$$

where $\mathbf{t} = \mathbf{e}_\theta$ is the tangent vector to $\mathbf{x}$ on SC_r. Moreover, the inequality (4.44) *holds with equality for all $\mathbf{x} \in \mathrm{SC}_r$ if and only if*

$$\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{n}}(\mathbf{x}) \cdot \frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{t}}(\mathbf{x}) = 0, \quad \textit{for all } \mathbf{x} \in \mathrm{SC}_r. \tag{4.45}$$

Proof. The result follows from a straightforward modification of the proof of [SS08b, Lemma 2.2] in the case $n = 2$ (by specifying the domain as SC_r, and applying the transformation $\boldsymbol{\varphi} \mapsto \frac{1}{\alpha}\boldsymbol{\varphi}$), noting that the Cauchy-Schwartz inequality

$$\left|\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{n}} \times \frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{t}}\right| \leq \left|\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{n}}\right| \left|\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{t}}\right|, \tag{4.46}$$

holds with equality if and only if (4.45) holds. □

The following lemma is a particular case of [SS08b, Lemma 2.3] when $n = 2$, after the transformation $t \mapsto \frac{t}{\alpha}$.

Lemma 4.2.2. *Define the function* $g_\alpha : (0,\infty) \to \mathbb{R}$ *by*

$$g_\alpha(t) = \left(t^2 + \frac{\alpha^4}{t^2}\right)^{\frac{1}{2}}, \quad t \in (0,\infty).$$

Then g_α *is convex on* $(0,\infty)$ *and strictly increasing on* $[\alpha,\infty)$.

Lemma 4.2.3. *Let* $\boldsymbol{\varphi} \in \mathcal{A}_\alpha$. *Then for each* $r \in (0,1)$,

$$\fint_{\mathrm{SC}_r} \left|\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{t}}(\mathbf{x})\right| \, \mathrm{d}l \geq \fint_{\mathrm{SC}_r} \left|\frac{\partial \boldsymbol{\varphi}_{DC_\alpha}}{\partial \mathbf{t}}(\mathbf{x})\right| \, \mathrm{d}l = \sqrt{2}\alpha, \tag{4.47}$$

where $\boldsymbol{\varphi}_{DC_\alpha}$ *is given by* (4.42). *Moreover, the above inequality holds with equality if and only if there exists* $\mathbf{k} \in C^1((0,1),\mathbb{R}^2)$ *and* $\Theta \in C^1(\Gamma,\mathbb{R})$ *such that*

$$\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{k}(r) + \frac{\alpha r}{\sqrt{2}} \begin{pmatrix} \cos(\Theta(r,\theta)) \\ \sin(\Theta(r,\theta)) \end{pmatrix}, \tag{4.48}$$

and

$$\Theta_\theta(r,\theta)\left(\mathbf{k}'(r) \cdot \begin{pmatrix} \cos(\Theta(r,\theta)) \\ \sin(\Theta(r,\theta)) \end{pmatrix} + \frac{\alpha}{\sqrt{2}}\right) = \sqrt{2}\alpha. \tag{4.49}$$

Proof. By (4.39), both $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}_{DC_\alpha}$ map Γ_r to a region with area $\frac{\pi}{2}(\alpha r)^2$. Moreover, by (4.38), the region is enclosed by $\boldsymbol{\varphi}(\mathrm{SC}_r)$. By the isoperimetric inequality, we have that the length of the perimeter of $\boldsymbol{\varphi}(\mathrm{SC}_r)$ is bounded below by the length of the perimeter of a circle with the same area, i.e.

$$\int_{\mathrm{SC}_r} \left|\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{t}}(\mathbf{x})\right| \, \mathrm{d}l \geq \sqrt{2}\pi\alpha r = \int_{\mathrm{SC}_r} \left|\frac{\partial \boldsymbol{\varphi}_{DC_\alpha}}{\partial \mathbf{t}}(\mathbf{x})\right| \, \mathrm{d}l, \tag{4.50}$$

which yields the first claim (4.47) (see Figure (4-2)). Furthermore, (4.50) holds with equality if and only if for each $r \in (0,1)$, $\boldsymbol{\varphi}(\mathrm{SC}_r)$ is a circle[3] with radius $\frac{\alpha r}{\sqrt{2}}$, and centre $\mathbf{k}(r)$ for some $\mathbf{k}(r) \in \mathbb{R}^2$. This is if and only if for all $r \in (0,1)$,

$$|\boldsymbol{\varphi}(\mathbf{x}) - \mathbf{k}(r)| = \frac{\alpha r}{\sqrt{2}}, \qquad \text{for all } \mathbf{x} \in \mathrm{SC}_r. \tag{4.51}$$

Since $\boldsymbol{\varphi} \in \mathcal{A}_\alpha$, (4.51) implies that $\mathbf{k} \in C^1(0,1)$. Now, define the functions $R \in C^1(\Gamma,\mathbb{R})$ and $\Theta \in C^1(\Gamma,\mathbb{R})$ to be such that

$$\boldsymbol{\varphi}(\mathbf{x}) - \mathbf{k}(r) = \begin{pmatrix} R(r,\theta)\cos(\Theta(r,\theta)) \\ R(r,\theta)\sin(\Theta(r,\theta)) \end{pmatrix}.$$

[3]Note that a circle with a perimeter of length $\sqrt{2}\pi\alpha r$ has radius $\frac{\alpha r}{\sqrt{2}}$, and area $\frac{\pi(\alpha r)^2}{2}$.

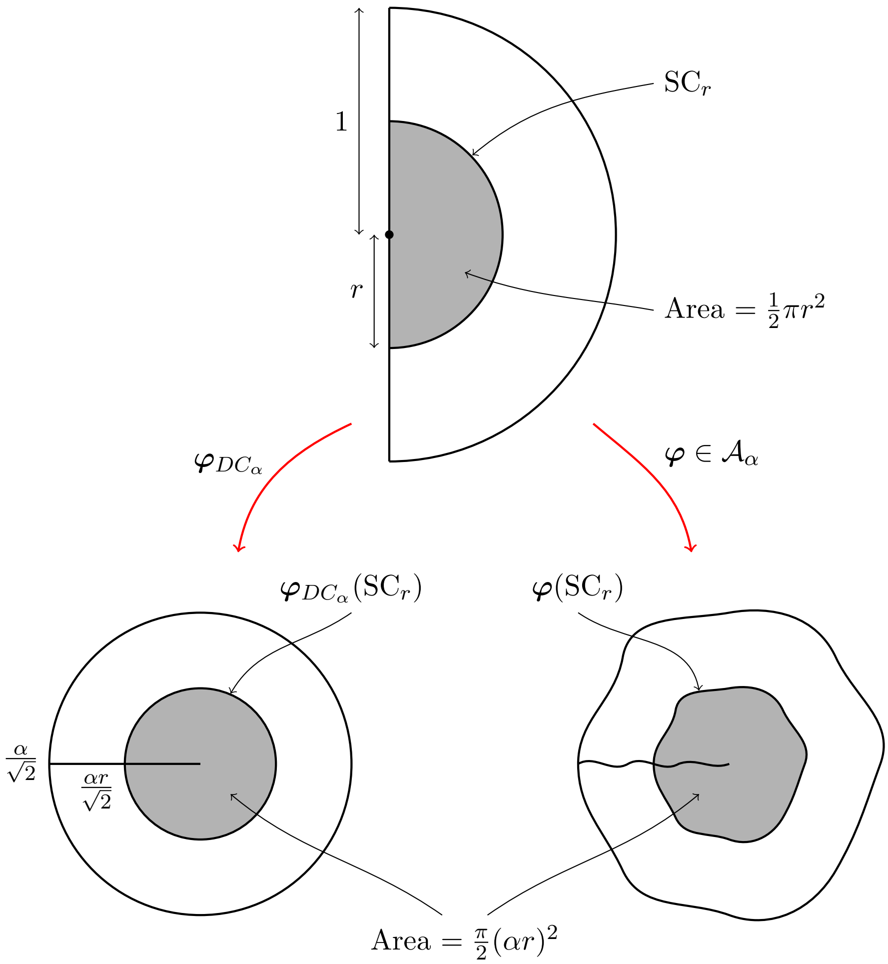


Figure 4-2: By the isoperimetric inequality, the smallest deformed perimeter of each semi-circle SC_r, under the constraint $\det \nabla\boldsymbol{\varphi} = \alpha^2$, is obtained when $\boldsymbol{\varphi}(\mathrm{SC}_r)$ is a circle.

Hence R, Θ are polar coordinates for the translated deformed configuration. This representation gives that the condition (4.51) is equivalent to $R(r,\theta)^2 = \frac{(\alpha r)^2}{2}$ for all $\mathbf{x} \in \mathrm{SC}_r$, and any $r \in (0,1)$. Hence, $R(r,\theta) \equiv R(r) = \frac{\alpha r}{\sqrt{2}}$. By Lemma 4.0.1, we also have that

$$\begin{aligned}
\nabla(\boldsymbol{\varphi} - \mathbf{k}) &= \mathbf{Q}(\Theta)\begin{pmatrix} R_r & \frac{R_\theta}{r} \\ R\Theta_r & \frac{R\Theta_\theta}{r} \end{pmatrix}\mathbf{Q}(\theta)^T \\
&= \mathbf{Q}(\Theta)\begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{\alpha r}{\sqrt{2}}\Theta_r & \frac{\alpha}{\sqrt{2}}\Theta_\theta \end{pmatrix}\mathbf{Q}(\theta)^T, \qquad (4.52)
\end{aligned}$$

hence by (4.52),

$$\nabla\boldsymbol{\varphi} = \left[\begin{pmatrix} k_1'(r) & 0 \\ k_2'(r) & 0 \end{pmatrix} + \mathbf{Q}(\Theta)\begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{\alpha r}{\sqrt{2}}\Theta_r & \frac{\alpha}{\sqrt{2}}\Theta_\theta \end{pmatrix}\right]\mathbf{Q}(\theta)^T \tag{4.53}$$

and since $\mathbf{Q}(t) \in SO(2)$ for all $t \in \mathbb{R}$, by (4.39),

$$\alpha^2 = \det(\nabla\boldsymbol{\varphi}) = \frac{\alpha}{\sqrt{2}}\Theta_\theta\left(k_1'(r)\cos(\Theta) + k_2'(r)\sin(\Theta) + \frac{\alpha}{\sqrt{2}}\right),$$

which gives (4.49). $\square$

If $\boldsymbol{\varphi}$ satisfies (4.39) alone, this is not enough for $\boldsymbol{\varphi}$ to simplify to $\boldsymbol{\varphi}_{DC_\alpha}$ uniquely. However, the following Lemma shows that this uniqueness property holds if we also specify $\boldsymbol{\varphi}$ to satisfy (4.45).

Lemma 4.2.4. *Let $\boldsymbol{\varphi} \in \mathcal{A}_\alpha$. Then $\boldsymbol{\varphi}$ is of the form* (4.48) *and satisfies* (4.45) *and* (4.49) *if and only if $\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_{DC_\alpha}$, where $\boldsymbol{\varphi}_{DC_\alpha}$ is given by* (4.42).

Proof. Sufficiency is clear since $\boldsymbol{\varphi}_{DC_\alpha}$ clearly takes the form (4.48), and

$$\nabla\boldsymbol{\varphi}_{DC_\alpha} = \mathbf{Q}(2\theta)\,\mathrm{diag}\left(\tfrac{\alpha}{\sqrt{2}}, \sqrt{2}\alpha\right)\mathbf{Q}(\theta)^T$$

means that (4.45) and (4.49) is satisfied.

For necessity, we take $\boldsymbol{\varphi}$ to be of the form (4.48), and calculate its deformation gradient with the aid of (4.53),

$$\nabla\boldsymbol{\varphi} = \left[\begin{pmatrix} k_1'(r) & 0 \\ k_2'(r) & 0 \end{pmatrix} + \mathbf{Q}(\Theta)\begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{\alpha r}{\sqrt{2}}\Theta_r & \frac{\alpha}{\sqrt{2}}\Theta_\theta \end{pmatrix}\right]\mathbf{Q}(\theta)^T$$

Then

$$\begin{aligned}
\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{n}} &= \nabla\boldsymbol{\varphi}\mathbf{e}_r = \left[\begin{pmatrix} k_1'(r) & 0 \\ k_2'(r) & 0 \end{pmatrix} + \mathbf{Q}(\Theta)\begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{\alpha r}{\sqrt{2}}\Theta_r & \frac{\alpha}{\sqrt{2}}\Theta_\theta \end{pmatrix}\right]\mathbf{e}_x \\
&= \mathbf{k}'(r) + \alpha\mathbf{Q}(\Theta)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{r}{\sqrt{2}}\Theta_r \end{pmatrix}, \\
\frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{t}} &= \left[\begin{pmatrix} k_1'(r) & 0 \\ k_2'(r) & 0 \end{pmatrix} + \mathbf{Q}(\Theta)\begin{pmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{\alpha r}{\sqrt{2}}\Theta_r & \frac{\alpha}{\sqrt{2}}\Theta_\theta \end{pmatrix}\right]\mathbf{e}_y \\
&= \frac{\alpha}{\sqrt{2}}\Theta_\theta\mathbf{Q}(\Theta)\mathbf{e}_y.
\end{aligned}$$

Hence by (4.45), we have that

$$\begin{pmatrix} -\sin(\Theta) \\ \cos(\Theta) \end{pmatrix} \cdot \mathbf{k}'(r) + \frac{\alpha r}{\sqrt{2}}\Theta_r = 0. \tag{4.54}$$

Now, by integrating (4.49) with respect to θ, we have

$$-\mathbf{k}'(r) \cdot \begin{pmatrix} -\sin(\Theta) \\ \cos(\Theta) \end{pmatrix} + \frac{\alpha}{\sqrt{2}}\Theta = \frac{\alpha}{\sqrt{2}}\left(2\theta + \xi(r)\right), \tag{4.55}$$

where $\xi : (0,1) \to \mathbb{R}$ is an arbitrary function. Combining (4.54) with (4.55) and rearranging then gives

$$\frac{\partial}{\partial r}(r\Theta) = 2\theta + \xi(r),$$

which through integrating with respect to r yields

$$\Theta(r,\theta) = 2\theta + \widehat{\xi}(r) + \frac{g(\theta)}{r}, \tag{4.56}$$

where $\widehat{\xi}$ is such that $r\widehat{\xi}(r)$ is the antiderivative of $\xi(r)$, and $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is an arbitrary function. Now, substitute (4.56) into (4.49) and rearrange to obtain

$$\left(2 + \frac{g'(\theta)}{r}\right)\mathbf{k}'(r) \cdot \begin{pmatrix} \cos(\Theta) \\ \sin(\Theta) \end{pmatrix} = -\frac{\alpha}{\sqrt{2}}\frac{g'(\theta)}{r}. \tag{4.57}$$

For any θ, assume for a contradiction that $g'(\theta) \neq 0$. Then by (4.57),

$$\lim_{r\to 0} \mathbf{k}'(r) \cdot \begin{pmatrix} \cos(\Theta) \\ \sin(\Theta) \end{pmatrix} = -\frac{\alpha}{\sqrt{2}}, \tag{4.58}$$

which is independent of θ. This implies that $\lim_{r\to 0}\Theta(r,\theta)$ is independent of θ also, which contradicts (4.56). Hence, we have that $g(\theta)$ is a constant function. Applying this with (4.57) gives

$$\mathbf{k}'(r) \cdot \begin{pmatrix} \cos(\Theta) \\ \sin(\Theta) \end{pmatrix} = 0, \quad \text{for all } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Hence,

$$\mathbf{k}'(r) = 0, \tag{4.59}$$

that is, $\mathbf{k}$ is constant. Therefore, by (4.54) and (4.55),

$$\Theta_r = 0, \tag{4.60}$$

$$\Theta = 2\theta + \xi(r). \tag{4.61}$$

Together (4.60) and (4.61) imply that

$$\Theta = 2\theta + c, \tag{4.62}$$

where c is a constant. Overall,

$$\boldsymbol{\varphi} = \mathbf{k} + \frac{\alpha r}{\sqrt{2}} \begin{pmatrix} \cos(2\theta + c) \\ \sin(2\theta + c) \end{pmatrix}.$$

The constraints (4.40) on $\boldsymbol{\varphi}$ imply that $\mathbf{k} = \mathbf{0}$ and (without loss of generality) $c = 0$. $\square$

Proposition 4.2.5. *Let $\boldsymbol{\varphi} \in \mathcal{A}_\alpha$, and let $r \in (0, 1)$. Then*

$$⨍_{\mathrm{SC}_r} |\nabla \boldsymbol{\varphi}(\mathbf{x})| \, \mathrm{d}l \geq ⨍_{\mathrm{SC}_r} |\nabla \boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x})| \, \mathrm{d}l, \tag{4.63}$$

where $\boldsymbol{\varphi}_{DC_\alpha}$ is given by (4.42)*. Moreover,* (4.63) *holds with equality if and only if $\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_{DC_\alpha}$.*

Proof. By Lemma 4.2.1, Lemma 4.2.2, Jensen's inequality, and Lemma 4.2.3 together with the fact that $g_\alpha(t)$ is strictly increasing for $t \geq \sqrt{2}\alpha$,

$$\begin{aligned} ⨍_{\mathrm{SC}_r} |\nabla \boldsymbol{\varphi}(\mathbf{x})| \, \mathrm{d}l &\geq ⨍_{\mathrm{SC}_r} g_\alpha \left(\left| \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{t}}(\mathbf{x}) \right| \right) \, \mathrm{d}l \\ &\geq g_\alpha \left(⨍_{\mathrm{SC}_r} \left| \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{t}}(\mathbf{x}) \right| \, \mathrm{d}l \right) \\ &\geq g_\alpha \left(⨍_{\mathrm{SC}_r} \left| \frac{\partial \boldsymbol{\varphi}_{DC_\alpha}}{\partial \mathbf{t}}(\mathbf{x}) \right| \, \mathrm{d}l \right) = \sqrt{\frac{5}{2}}\alpha = ⨍_{\mathrm{SC}_r} |\nabla \boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x})| \, \mathrm{d}l, \end{aligned}$$

which gives us the desired inequality (4.63).

Suppose now (4.63) held with equality. Then each inequality above must be an equality, so by Lemma 4.2.3, since g_α is strictly increasing, $\boldsymbol{\varphi}$ is of the form (4.48) and satisfies (4.49). Furthermore, by Lemma 4.2.1 we have that (4.45) holds, so by Lemma 4.2.4, $\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_{DC_\alpha}$. $\square$

For any incompressible, isotropic material in two dimensions, one may write the stored energy function $W^{\mathrm{inc}} : M_1^{2\times 2} \to \mathbb{R}$ as a function $\sigma : (0, \infty) \to \mathbb{R}$ of the principal invariant $I_1 = |\mathbf{F}|^2$, namely

$$W^{\mathrm{inc}}(\mathbf{F}) = \sigma(|\mathbf{F}|) \qquad \mathbf{F} \in M_1^{2\times 2}. \tag{4.64}$$

When W is *polyconvex* (see [Bal77]), σ is convex. The following result applies to functions W such that (4.64) holds, where σ is convex and monotone increasing.

Theorem 4.2.6. *Let $\sigma : (0,\infty) \to \mathbb{R}$ be a convex and monotone increasing function satisfying* (4.64)*, and let $\boldsymbol{\varphi} \in \mathcal{A}_\alpha$. Then*

$$\int_\Gamma \sigma(|\nabla \boldsymbol{\varphi}(\mathbf{x})|)\, \mathrm{d}\mathbf{x} \geq \int_\Gamma \sigma(|\nabla \boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x})|)\, \mathrm{d}\mathbf{x}. \tag{4.65}$$

Moreover, if σ is strictly increasing, then (4.65) *holds with equality if and only if $\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_{DC_\alpha}$.*

Proof. By the convexity of σ, Jensen's inequality, and Proposition 4.2.5,

$$\begin{aligned}
\int_0^1 \int_{\mathrm{SC}_r} \sigma(|\nabla \boldsymbol{\varphi}(\mathbf{x})|)\, \mathrm{d}l \mathrm{d}r &\geq \int_0^1 \pi r\ \sigma\left(⨍_{\mathrm{SC}_r} |\nabla \boldsymbol{\varphi}(\mathbf{x})|\, \mathrm{d}l \right)\, \mathrm{d}r \\
&\geq \int_0^1 \pi r\ \sigma\left(⨍_{\mathrm{SC}_r} |\nabla \boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x})|\, \mathrm{d}l \right)\, \mathrm{d}r \\
&= \int_0^1 \int_{\mathrm{SC}_r} \sigma(|\nabla \boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x})|)\, \mathrm{d}l \mathrm{d}r.
\end{aligned}$$

Hence the inequality (4.65) holds. Furthermore, if all of the above inequalities hold with equality, then by monotonicity of σ, we have that (4.63) holds with equality. By Proposition 4.2.5 this is if and only if $\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_{DC_\alpha}$. □

The following straightforward corollary of Theorem 4.2.6 (by setting $\alpha = 1$) shows that the two-dimensional, incompressible double-covering map is the unique global minimiser of (4.43) over $\mathcal{A}_1$.

Corollary 4.2.7. *Let E^{inc} be given by* (4.43)*, let $\sigma : (0,\infty) \to \mathbb{R}$ be a convex and monotone increasing function satisfying* (4.64)*, and let $\boldsymbol{\varphi} \in \mathcal{A}_1$ (given by* (4.40) *with $\alpha = 1$). Then*

$$E^{\mathrm{inc}}[\boldsymbol{\varphi}] \geq E^{\mathrm{inc}}[\boldsymbol{\varphi}_{DC_1}]. \tag{4.66}$$

Moreover, if σ is strictly increasing, then (4.66) *holds with equality if and only if $\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_{DC_1}$.*

4.2.2 The three dimensional case

To extend our results to three dimensions, we let Ω be the region defined by (4.15), and consider *incompressible* deformations $\boldsymbol{\varphi}$ on Ω subject to the forced self-contact boundary condition on Γ_2 given by

$$\boldsymbol{\varphi}(0, x_2, x_3) = \boldsymbol{\varphi}(0, -x_2, x_3), \quad \text{for all } 0 < x_2 < 1,\ 0 < x_3 < L, \tag{4.67}$$

and a slip boundary condition on Γ_3 given by

$$\boldsymbol{\varphi}(x_1, x_2, 0) = 0, \quad \boldsymbol{\varphi}(x_1, x_2, L) = \gamma L, \quad \text{for all } (x_1, x_2) \in \Gamma, \tag{4.68}$$

for some given $\gamma > 0$. To eliminate nonuniqueness, we impose the additional constraints,

$$\int_\Omega \varphi_1(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_\Omega \varphi_2(\mathbf{x})\ \mathrm{d}\mathbf{x} = 0, \tag{4.69a}$$

$$\int_\Omega \frac{\partial \varphi_1}{\partial x_2}(\mathbf{x})\ \mathrm{d}\mathbf{x} = 0 < \int_\Omega \frac{\partial \varphi_2}{\partial x_1}(\mathbf{x})\ \mathrm{d}\mathbf{x}. \tag{4.69b}$$

A straightforward extension of the two dimensional problem arises from only considering deformations of the form[4]

$$\boldsymbol{\varphi}(\mathbf{x}) = \begin{pmatrix} \varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \\ \gamma x_3 \end{pmatrix}. \tag{4.70}$$

Define the set of admissible deformations on Ω by

$$\begin{aligned} \widetilde{\mathcal{A}} = \{\boldsymbol{\varphi} \in C^1(\Omega, \mathbb{R}^3) \mid \det(\nabla \boldsymbol{\varphi}) = 1,\ \boldsymbol{\varphi} \text{ satisfies (4.67), (4.68), (4.69),} \\ \text{and } \boldsymbol{\varphi} \text{ is of the form (4.70)}\}. \end{aligned} \tag{4.71}$$

Define

$$\widetilde{\boldsymbol{\varphi}}_{DC\gamma}(\mathbf{x}) = \begin{pmatrix} \frac{1}{\sqrt{2\gamma(x_1^2+x_2^2)}}(x_1^2 - x_2^2) \\ \frac{1}{\sqrt{2\gamma(x_1^2+x_2^2)}} 2x_1x_2 \\ \gamma x_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\varphi}_{DC_\alpha}(x_1, x_2) \\ \gamma x_3 \end{pmatrix}, \tag{4.72}$$

[4]Note that if we considered the slightly more general case $\varphi_3(\mathbf{x}) = \zeta(x_3)$, the fact that $\boldsymbol{\varphi}$ is incompressible means that $\zeta'(x_3)$ must be constant, and thus $\zeta(x_3) = \gamma x_3$ everywhere by (4.68).

where $\alpha = \frac{1}{\sqrt{\gamma}}$. Note that $\widetilde{\boldsymbol{\varphi}}_{DC\gamma} \in \widetilde{\mathcal{A}}$, but $\widetilde{\boldsymbol{\varphi}}_{DC\gamma} \notin C^1(\overline{\Omega}, \mathbb{R}^3)$. We aim to show that the unique global minimiser of the stored energy

$$E[\boldsymbol{\varphi}] = \int_\Omega W^{\mathrm{inc}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \tag{4.73}$$

over the set $\widetilde{\mathcal{A}}$ is the double-covering map $\widetilde{\boldsymbol{\varphi}}_{DC\gamma}$, for a large class of functions W^{inc}. Define the projection $P : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$P\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Note that P has the property that $P\widetilde{\boldsymbol{\varphi}}_{DC\gamma} = \boldsymbol{\varphi}_{DC_\alpha}$. Then for any $\boldsymbol{\varphi} \in \widetilde{\mathcal{A}}$, we have that $P\boldsymbol{\varphi} \in \mathcal{A}_\alpha$, and

$$1 = \det(\nabla\boldsymbol{\varphi}(\mathbf{x})) = \gamma\left(\frac{\partial\varphi_1}{\partial x_1}\frac{\partial\varphi_2}{\partial x_2} - \frac{\partial\varphi_1}{\partial x_2}\frac{\partial\varphi_2}{\partial x_1}\right) = \gamma\det(\nabla P\boldsymbol{\varphi}(\mathbf{x})), \tag{4.74}$$

$$|\nabla\boldsymbol{\varphi}(\mathbf{x})|^2 = \left(\frac{\partial\varphi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi_1}{\partial x_2}\right)^2 + \left(\frac{\partial\varphi_2}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi_2}{\partial x_2}\right)^2 + \gamma^2 = |\nabla P\boldsymbol{\varphi}(\mathbf{x})|^2 + \gamma^2, \tag{4.75}$$

$$|\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x})|^2 = \left|\begin{pmatrix} \gamma\varphi_{2,2} & \gamma\varphi_{2,1} & 0 \\ \gamma\varphi_{1,2} & \gamma\varphi_{1,1} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{pmatrix}\right|^2 = \gamma^2|\nabla P\boldsymbol{\varphi}(\mathbf{x})|^2 + \frac{1}{\gamma^2}. \tag{4.76}$$

Note that given any incompressible, isotropic stored energy function W^{inc}, we may write W^{inc} as a function of the principal invariants $I_1 = |\mathbf{F}|^2$ and $I_2 = |\mathrm{Cof}\mathbf{F}|^2$, i.e.

$$W^{\mathrm{inc}}(\mathbf{F}) = h^{\mathrm{inc}}(|\mathbf{F}|, |\mathrm{Cof}\mathbf{F}|), \qquad \mathbf{F} \in M_1^{3\times3}. \tag{4.77}$$

If W^{inc} is polyconvex, then h^{inc} is convex. By considering h^{inc} convex and monotone increasing in each argument, we have the following theorem.

Theorem 4.2.8. *Let $h^{\mathrm{inc}} : (0,\infty)^2 \to \mathbb{R}$ be a convex and monotone increasing function in both arguments that satisfies* (4.77)*, and let $\boldsymbol{\varphi} \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is given by* (4.71)*. Then*

$$\begin{aligned} E^{\mathrm{inc}}[\boldsymbol{\varphi}] &= \int_\Omega h^{\mathrm{inc}}(|\nabla\boldsymbol{\varphi}(\mathbf{x})|, |\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x})|)\,\mathrm{d}\mathbf{x} \\ &\geq \int_\Omega h^{\mathrm{inc}}(|\nabla\widetilde{\boldsymbol{\varphi}}_{DC\gamma}(\mathbf{x})|, |\mathrm{Cof}\nabla\widetilde{\boldsymbol{\varphi}}_{DC\gamma}(\mathbf{x})|)\,\mathrm{d}\mathbf{x} = E^{\mathrm{inc}}[\widetilde{\boldsymbol{\varphi}}_{DC\gamma}], \end{aligned}$$

where $\widetilde{\boldsymbol{\varphi}}_{DC\gamma}$ is given by (4.72)*. Moreover, if h^{inc} is strictly increasing, the above holds with equality if and only if $\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}}_{DC\gamma}$.*

Proof. Note that the function $\rho : (0,\infty) \to \mathbb{R}$ defined by

$$\rho(t) = h^{\mathrm{inc}}\left(\sqrt{t^2+\gamma^2}, \sqrt{(\gamma t)^2 + \frac{1}{\gamma^2}}\right)$$

is convex and monotone increasing on $(0,\infty)$, since the mappings $t \mapsto \sqrt{t^2+\gamma^2}$, $t \mapsto \sqrt{(\gamma t)^2 + \frac{1}{\gamma^2}}$, and $t \mapsto h^{\mathrm{inc}}(t)$ are convex and monotone increasing. By (4.75) and (4.76), for any $\boldsymbol{\varphi} \in \widetilde{\mathcal{A}}$, we have that

$$h^{\mathrm{inc}}\left(|\nabla\boldsymbol{\varphi}|, |\mathrm{Cof}\nabla\boldsymbol{\varphi}|\right) = \rho(|\nabla P\boldsymbol{\varphi}|).$$

Then

$$\begin{aligned} E[\boldsymbol{\varphi}] &= \int_\Omega h^{\mathrm{inc}}(|\nabla\boldsymbol{\varphi}(\mathbf{x})|, |\mathrm{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x})|)\,\mathrm{d}\mathbf{x} \\ &= L\int_\Gamma \rho(|\nabla P\boldsymbol{\varphi}(\mathbf{x})|)\,\mathrm{d}S(\mathbf{x}). \end{aligned} \tag{4.78}$$

By Theorem 4.2.6, since ρ is convex and monotone increasing, and $P\boldsymbol{\varphi} \in \mathcal{A}_\alpha$ where $\alpha = \frac{1}{\sqrt{\gamma}}$,

$$\int_\Gamma \rho(|\nabla P\boldsymbol{\varphi}(\mathbf{x})|)\,\mathrm{d}S(\mathbf{x}) \geq \int_\Gamma \rho(|\nabla\boldsymbol{\varphi}_{DC_\alpha}(x_1,x_2)|)\,\mathrm{d}S(\mathbf{x}), \tag{4.79}$$

where $\boldsymbol{\varphi}_{DC_\alpha}$ is given by (4.42). Hence by (4.78) and (4.79),

$$E[\boldsymbol{\varphi}] \geq L\int_\Gamma \rho(|\nabla\boldsymbol{\varphi}_{DC_\alpha}(x_1,x_2)|)\,\mathrm{d}S(\mathbf{x}) = E[\widetilde{\boldsymbol{\varphi}}_{DC\gamma}].$$

This holds with equality if and only if $P\boldsymbol{\varphi} = \boldsymbol{\varphi}_{DC_\alpha}$. Since $\boldsymbol{\varphi} \in \widetilde{\mathcal{A}}$, this is if and only if $\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}}_{DC\gamma}$. $\square$

Remark 4.2.9. The slip boundary condition (4.68) gives rise to a natural boundary condition for a minimiser where the stress acting on Γ at $x_3 = 0$ or L has zero tangential component. This condition is indeed met for when $\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}}_{DC\gamma}$, since by Lemma 4.0.2,

$$\begin{aligned} \widehat{\mathbf{S}}(\nabla\widetilde{\boldsymbol{\varphi}}_{DC\gamma})\mathbf{n} &= \pm\widehat{\mathbf{S}}\left(\mathbf{Q}(2\theta)\,\mathrm{diag}\left(\tfrac{1}{\sqrt{2\gamma}}, \sqrt{\tfrac{2}{\gamma}}, \gamma\right)\mathbf{Q}(\theta)^T\right)\mathbf{e}_3 \\ &= \pm\mathbf{Q}(2\theta)\,\mathrm{diag}(\Phi_1,\Phi_2,\Phi_3)\mathbf{e}_3 = \pm\Phi_3\mathbf{e}_3, \end{aligned}$$

where $\Phi_i = \Phi_{,i}\left(\frac{1}{\sqrt{2\gamma}}, \sqrt{\frac{2}{\gamma}}, \gamma\right)$.

It is of interest to determine whether $\widetilde{\boldsymbol{\varphi}}_{DC\alpha}$ is in fact the global minimiser of E^{inc}

(for stored energy functions of this class) among deformations of the general form

$$\boldsymbol{\varphi}(\mathbf{x}) = \begin{pmatrix} \varphi_1(\mathbf{x}) \\ \varphi_2(\mathbf{x}) \\ \varphi_3(\mathbf{x}) \end{pmatrix}.$$

The strategy used for Theorem 4.2.6 does not directly apply to such deformations, since the deformed area of any half-disk Γ_r is not necessarily equal to that of $\widetilde{\boldsymbol{\varphi}}_{DC\alpha}(\Gamma_r)$. The problem only needs to be reduced to considering $\boldsymbol{\varphi} \in \widetilde{\mathcal{A}}$.

Chapter 5

Explicit stored energy comparisons

Following from Remark 1.2.29, in this chapter we seek an example of a (pure) homogeneous deformation which fails the quasiconvexity at the boundary condition (1.37), but remains a weak local minimiser. We will compare the stored energy of a particular 'crease-like' deformation with that of a corresponding pure homogeneous deformation.

In Section 5.1, we take a compressible neo-Hookean material occupying a rectangular region in two dimensions in its reference state, subject to a displacement boundary condition on its sides and bottom that matches that of a pure homogeneous deformation. The top part of the boundary is left free and hence a crease may develop on this surface. We construct a deformation that causes self-contact on the deformed top surface, and compare its total stored energy with that of the unique pure homogeneous deformation which satisfies the given displacement boundary condition and leaves the upper boundary stress free. We will find conditions on the stretches of the given pure homogeneous deformation and the dimensions of the block for which the compressible neo-Hookean stored energy of the crease-like deformation is lower than that of the pure homogeneous deformation.

In Section 5.2, we take a compressible neo-Hookean material occupying the half-space in two dimensions in its reference state. We suppose that admissible deformations must have pure homogeneous behaviour at infinity, i.e. $\boldsymbol{\varphi} \sim \boldsymbol{\varphi}^h$ for $|\mathbf{x}| \gg 1$.[1] We will consider a deformation constructed with conformal maps in the complex half-plane to develop a sulcus on the free surface, and compose this with the appropriate pure homogeneous deformation to satisfy the boundary condition at infinity. We will show that it is possible for the Dirichlet part of the stored energy to be lower for this crease-

[1] We will also call this the 'far-field' behaviour of $\boldsymbol{\varphi}$.

like deformation than for the pure homogeneous deformation.

5.1 A piecewise continuous crease-like deformation

In two dimensions, we consider a block of material occupying the region $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid -2L < x_1 < 2L,\ 0 < x_2 < 2\}$, partitioned in the following way:

$$\begin{aligned}
\Omega_1^{\pm} &= \{\mathbf{x} \in \mathbb{R}^2 \mid L < \pm x_1 < 2L,\ 0 < x_2 < 1\} \\
\Omega_2^{\pm} &= \{\mathbf{x} \in \mathbb{R}^2 \mid L < \pm x_1 < 2L,\ 1 < x_2 < 2\} \\
\Omega_3^{\pm} &= \left\{\mathbf{x} \in \mathbb{R}^2 \mid 0 < \pm x_1 < L,\ 0 < x_2 < \frac{2L}{L+|x_1|}\right\} \\
\Omega_4^{\pm} &= \left\{\mathbf{x} \in \mathbb{R}^2 \mid 0 < \pm x_1 < L,\ \frac{2L}{L+|x_1|} < x_2 < 2\right\},
\end{aligned}$$

see Figure 5-1.

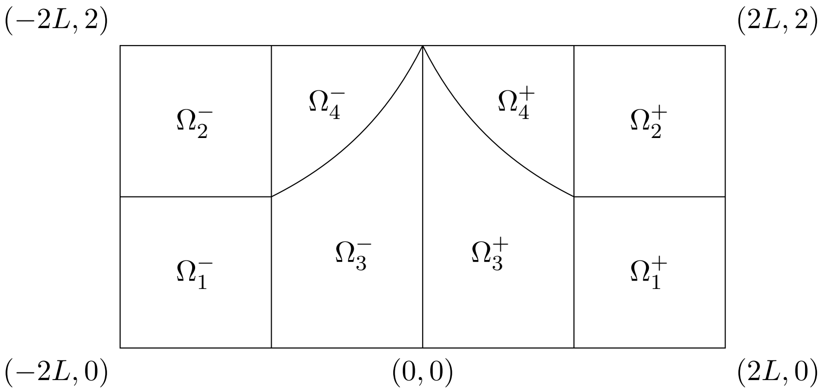


Figure 5-1: The partition $\Omega = \bigcup_{i=1}^4 \Omega_i^+ \cup \Omega_i^-$.

Consider the piecewise continuous deformation given by

$$\boldsymbol{\varphi}^p(\mathbf{x}) = \begin{cases} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_1^{\pm} \\ \begin{pmatrix} \frac{2x_1}{|x_1|}\left(L + \frac{|x_1|-2L}{3-x_2}\right) \\ x_2 \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_2^{\pm} \\ \begin{pmatrix} x_1 \\ \frac{x_2}{2}\left(1 + \frac{|x_1|}{L}\right) \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_3^{\pm} \\ \begin{pmatrix} \frac{2x_1}{|x_1|}\left(L + 2\frac{|x_1|-2L}{6-x_2(1+\frac{|x_1|}{L})}\right) \\ \frac{x_2}{2}\left(1 + \frac{|x_1|}{L}\right) \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_4^{\pm}. \end{cases}$$

This deformation has the property that

$$\begin{aligned} \boldsymbol{\varphi}^p(\Omega_1^{\pm}) &= \Omega_1^{\pm}, \\ \boldsymbol{\varphi}^p(\Omega_2^{\pm} \cup \Omega_4^{\pm}) &= \{\mathbf{x} \in \mathbb{R}^2 \mid 0 < \pm x_1 < 2L,\ 1 < x_2 < 2\} \text{ a.e.} \\ \boldsymbol{\varphi}^p(\Omega_3^{\pm}) &= \{\mathbf{x} \in \Omega \mid 0 < \pm x_1 < L,\ 0 < x_2 < 1, \}, \end{aligned}$$

see Figure 5-2.

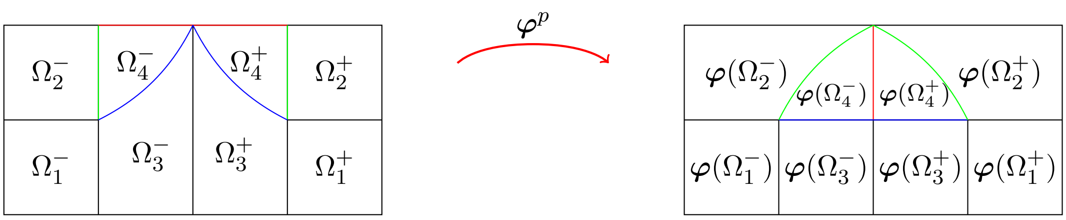


Figure 5-2: The boundary shaded red is mapped by $\boldsymbol{\varphi}$ to itself.

Define

$$\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\varphi}^h(\boldsymbol{\varphi}^p(\mathbf{x})), \tag{5.1}$$

and $f(x_1, x_2) = 6 - x_2\left(1 + \frac{|x_1|}{L}\right)$. The deformation gradient of $\boldsymbol{\varphi}$ is given by

$$\nabla\boldsymbol{\varphi}(\mathbf{x}) = \begin{cases} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_1^{\pm} \\ \begin{pmatrix} \frac{2\lambda_1}{3-x_2} & \frac{2\lambda_1 x_1}{|x_1|}\left(\frac{|x_1|-2L}{(3-x_2)^2}\right) \\ 0 & \lambda_2 \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_2^{\pm} \\ \begin{pmatrix} \lambda_1 & 0 \\ \frac{\lambda_2 x_2}{2}\left(\frac{x_1}{L|x_1|}\right) & \frac{\lambda_2}{2}\left(1 + \frac{|x_1|}{L}\right) \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_3^{\pm} \\ \begin{pmatrix} \frac{4\lambda_1(6-3x_2)}{f(x_1,x_2)^2} & \frac{4\lambda_1 x_1}{|x_1|}\left(\frac{(|x_1|-2L)\left(1+\frac{|x_1|}{L}\right)}{f(x_1,x_2)^2}\right) \\ \frac{\lambda_2 x_2}{2}\left(\frac{x_1}{L|x_1|}\right) & \frac{\lambda_2}{2}\left(1 + \frac{|x_1|}{L}\right) \end{pmatrix} & \text{if } \mathbf{x} \in \Omega_4^{\pm}. \end{cases}$$

For any isotropic material, the stored energy will be a function of $I_1(\boldsymbol{\varphi}(\mathbf{x})) = |\nabla\boldsymbol{\varphi}(\mathbf{x})|^2$, and $I_3(\boldsymbol{\varphi}(\mathbf{x})) = \det(\nabla\boldsymbol{\varphi}(\mathbf{x}))^2$. Hence,

$$I_1(\boldsymbol{\varphi}(\mathbf{x})) = \begin{cases} \lambda_1^2 + \lambda_2^2 & \text{if } \mathbf{x} \in \Omega_1^{\pm} \\ \frac{4\lambda_1^2}{(3-x_2)^2} + \frac{4\lambda_1^2(|x_1|-2L)^2}{(3-x_2)^4} + \lambda_2^2 & \text{if } \mathbf{x} \in \Omega_2^{\pm} \\ \lambda_1^2 + \left(\frac{\lambda_2 x_2}{2L}\right)^2 + \frac{\lambda_2^2}{4}\left(1 + \frac{|x_1|}{L}\right)^2 & \text{if } \mathbf{x} \in \Omega_3^{\pm} \\ 16\lambda_1^2\left(\frac{9(2-x_2)^2+(|x_1|-2L)^2\left(1+\frac{|x_1|}{L}\right)^2}{f(x_1,x_2)^4}\right) + \frac{\lambda_2^2}{4}\left(1 + \frac{x_2^2}{L^2} + \frac{|x_1|}{L}\right)^2 & \text{if } \mathbf{x} \in \Omega_4^{\pm}. \end{cases} \tag{5.2}$$

$$\sqrt{I_3(\boldsymbol{\varphi}(\mathbf{x}))} = \begin{cases} \lambda_1\lambda_2 & \text{if } \mathbf{x} \in \Omega_1^{\pm} \\ \frac{2\lambda_1\lambda_2}{(3-x_2)} & \text{if } \mathbf{x} \in \Omega_2^{\pm} \\ \frac{\lambda_1\lambda_2}{2}\left(1 + \frac{|x_1|}{L}\right) & \text{if } \mathbf{x} \in \Omega_3^{\pm} \\ \frac{2\lambda_1\lambda_2}{f(x_1,x_2)}\left(1 + \frac{|x_1|}{L}\right) & \text{if } \mathbf{x} \in \Omega_4^{\pm}. \end{cases} \tag{5.3}$$

Our goal is to compare the stored energy of a material undergoing this deformation with that of a pure homogeneous deformation given by

$$\boldsymbol{\varphi}^h(\mathbf{x}) = \mathbf{D}\mathbf{x}, \tag{5.4}$$

where

$$\mathbf{D} = \text{diag}(\lambda_1, \lambda_2).$$

5.1.1 Dirichlet energy

Define the constants

$$a = \frac{77}{12} - \frac{2}{9}\log(2) > 0 \tag{5.5a}$$

$$b = -\frac{59}{36} + 4\log(2) > 0 \tag{5.5b}$$

$$c = -\frac{5}{3} \tag{5.5c}$$

$$d = \frac{4}{3}. \tag{5.5d}$$

Lemma 5.1.1. *Let $L > 0$, let $\boldsymbol{\varphi}$ be given by* (5.1)*, and let $\boldsymbol{\varphi}^h$ be a pure homogeneous deformation given by* (5.4)*. Then*

$$\int_\Omega |\nabla \boldsymbol{\varphi}(\mathbf{x})|^2 \; \mathrm{d}\mathbf{x} < \int_\Omega |\nabla \boldsymbol{\varphi}^h(\mathbf{x})|^2 \; \mathrm{d}\mathbf{x}$$

if and only if $L > \sqrt{\frac{4}{5}}$, and

$$r^2 > -\frac{b}{c}L^2 + \frac{bd - ac}{c^2} - \frac{bd^2 - ac^2}{cL^2 + d}, \tag{5.6}$$

where $r = \frac{\lambda_2}{\lambda_1}$, and a, b, c, and d are given by (5.5).

Proof. We have, by symmetry,

$$\begin{aligned}
\int_\Omega I_1(\boldsymbol{\varphi}(\mathbf{x}))\mathrm{d}x_1 \; \mathrm{d}x_2 &= 2\int_{\Omega_1^+ \cup \Omega_2^+ \cup \Omega_3^+ \cup \Omega_4^+} I_1(\boldsymbol{\varphi}(\mathbf{x})) \; \mathrm{d}x_1 \mathrm{d}x_2 \\
&= 2\bigg(\int_{\Omega_1^+} I_1(\boldsymbol{\varphi}(\mathbf{x})) \; \mathrm{d}x_1 \mathrm{d}x_2 + \int_{\Omega_2^+} I_1(\boldsymbol{\varphi}(\mathbf{x}))\mathrm{d}x_1 \mathrm{d}x_2 \\
&\quad + \int_{\Omega_3^+} I_1(\boldsymbol{\varphi}(\mathbf{x})) \; \mathrm{d}x_1 \mathrm{d}x_2 + \int_{\Omega_4^+} I_1(\boldsymbol{\varphi}(\mathbf{x}))\mathrm{d}x_1 \mathrm{d}x_2 \bigg)
\end{aligned}$$

and by (5.2),

$$
\begin{aligned}
\int_{\Omega_1^+} I_1(\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}x_1\mathrm{d}x_2 &= (\lambda_1^2+\lambda_2^2)L \\
\int_{\Omega_2^+} I_1(\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}x_1\mathrm{d}x_2 &= \left(2\lambda_1^2+\lambda_2^2\right)L + \frac{7\lambda_1^2}{18}L^3 \\
\int_{\Omega_3^+} I_1(\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}x_1\mathrm{d}x_2 &= \left(2\log(2)\lambda_1^2+\frac{3}{4}\lambda_2^2\right)L + \frac{\lambda_2^2}{4L} \\
\int_{\Omega_4^+} I_1(\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}x_1\mathrm{d}x_2 &= \lambda_1^2\left[\left(\frac{101}{24}-\frac{19}{9}\log(2)\right)L + \left(-\frac{29}{24}+2\log(2)\right)L^3\right] \\
&\qquad + \frac{5\lambda_2^2}{12}\left(L+\frac{1}{L}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{\Omega} I_1(\boldsymbol{\varphi}(\mathbf{x}))\mathrm{d}x_1\mathrm{d}x_2 = \lambda_1^2\left[\left(\frac{173}{12}-\frac{2}{9}\log(2)\right)L + \left(-\frac{59}{36}+4\log(2)\right)L^3\right] \\
+ \frac{\lambda_2^2}{3}\left[19L+\frac{4}{L}\right].
\end{aligned}
$$

Comparing this with $\boldsymbol{\varphi}^h$, given the fact that $I_1(\boldsymbol{\varphi}^h(\mathbf{x})) = \lambda_1^2+\lambda_2^2$,

$$
\begin{aligned}
\int_{\Omega} I_1(\boldsymbol{\varphi}) - I_1(\boldsymbol{\varphi}^h)\ \mathrm{d}x_1\mathrm{d}x_2 &= \lambda_1^2\left[\left(\frac{77}{12}-\frac{2}{9}\log(2)\right)L + \left(-\frac{59}{36}+4\log(2)\right)L^3\right] \\
&\qquad + \frac{\lambda_2^2}{3}\left[-5L+\frac{4}{L}\right] \\
&= \lambda_1^2\left(aL+bL^3\right) + \lambda_2^2\left(cL+\frac{d}{L}\right).
\end{aligned}
$$

Therefore, for $\lambda_1, \lambda_2, L > 0$, we have by Lemma A.5.1[2] that

$$
\int_{\Omega} I_1(\boldsymbol{\varphi}(\mathbf{x})) - I_1(\boldsymbol{\varphi}^h(\mathbf{x}))\ \mathrm{d}x_1\mathrm{d}x_2 < 0
$$

if and only if $L > \sqrt{\frac{4}{5}} \approx 0.8944$, and (5.6) holds. $\square$

This gives us a condition on the compression ratio and dimensions of the block for when the Dirichlet (I_1) energy is smaller for the creaselike deformation $\boldsymbol{\varphi}$ than it is for pure compression, $\boldsymbol{\varphi}^h$. Given any $L > \sqrt{\frac{4}{5}}$, there exists a sufficiently large r such that (5.6) holds, and consequently a crease is (Dirichlet) energetically favourable. See Figure (5-3).

[2]We have given a proof for this claim in the Appendix due to its purely technical nature.

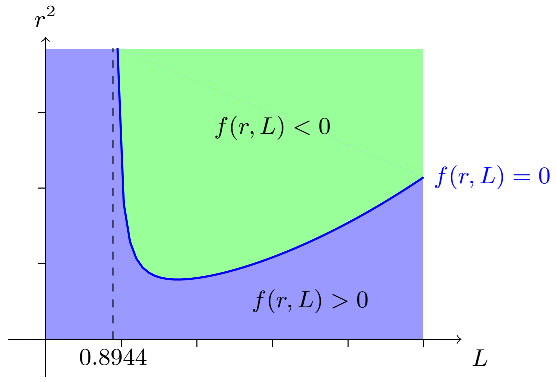


Figure 5-3: The regions for which $f(r,L)<0$ (green) or $f(r,L)>0$ (blue), where $r=\frac{\lambda_2}{\lambda_1}$ and $f(r,L)=aL+bL^3+r^2\left(cL+\frac{d}{L}\right)$.

Remark 5.1.2. Contrary to the necessary conditions for weak local minimality displayed in Chapter 2 (and 3), we have here a sufficient condition for when $\boldsymbol{\varphi}^h$ ceases to be a global minimiser of the Dirichlet energy *which depends on the dimensions of the considered domain*. The phenomenon of an absence of length scale in the study of the complementing condition or Agmon's condition does not occur in the present work.

5.1.2 Compressible neo-Hookean stored energy

The deformation $\boldsymbol{\varphi}$ given by (5.1) is not incompressible, so we must also consider the change in energy due to volume change. A standard model for the total stored energy is that of a compressible neo-Hookean material,

$$E[\mathbf{u}]=\int_\Omega \frac{1}{2}|\nabla\mathbf{u}(\mathbf{x})|^2+H(\det(\nabla\mathbf{u}(\mathbf{x})))\ \mathrm{d}\mathbf{x}, \tag{5.7}$$

The change in stored energy due to volume is represented here by the model function H. For many materials, such as those with a quasiconvex stored energy function [Bal77], H must be convex, in which case we have the following lemma.

Lemma 5.1.3. *Let $\boldsymbol{\varphi}$ be given by* (5.1)*, let $\boldsymbol{\varphi}^h$ be a pure homogeneous deformation given by* (5.4)*, and let $H:(0,\infty)\to\mathbb{R}$ be convex. Then*

$$\int_\Omega H(\det(\nabla\boldsymbol{\varphi}(\mathbf{x})))\ \mathrm{d}\mathbf{x}\geq\int_\Omega H(\det(\nabla\boldsymbol{\varphi}^h(\mathbf{x})))\ \mathrm{d}\mathbf{x}.$$

Proof. It can be shown through direct calculation, or by noting that $\boldsymbol{\varphi}$ maps Ω to the same region as $\boldsymbol{\varphi}^h$, that $\int_\Omega I_3(\boldsymbol{\varphi}(\mathbf{x}))\ \mathrm{d}\mathbf{x}=\int_\Omega I_3(\boldsymbol{\varphi}^h(\mathbf{x}))\ \mathrm{d}\mathbf{x}$. Therefore, by virtue of

Jensen's inequality, any convex choice of H yields

$$\begin{aligned}
\fint_\Omega H(I_3(\boldsymbol{\varphi}))\ \mathrm{d}\mathbf{x} &\geq H\left(\fint_\Omega I_3(\boldsymbol{\varphi})\ \mathrm{d}\mathbf{x}\right) \\
&= H\left(\fint_\Omega I_3(\boldsymbol{\varphi}^h)\ \mathrm{d}\mathbf{x}\right) \\
&= \fint_\Omega H(I_3(\boldsymbol{\varphi}^h))\ \mathrm{d}\mathbf{x},
\end{aligned}$$

since $\det(\nabla\boldsymbol{\varphi}^h) = \lambda_1\lambda_2$ is constant. The result follows by multiplying through by $|\Omega|$. □

Since H is an arbitrary convex function, explicitly integrating $H(\det(\nabla\boldsymbol{\varphi}(\mathbf{x})))$ over Ω in general cannot be carried out through standard methods, so we will consider a simple example. Let $H : (0,\infty) \to \mathbb{R}$ be the convex function defined by

$$H(\eta) = a_2\eta^2 + \frac{a_{-1}}{\eta}, \tag{5.8}$$

where $a_2 > 0$ and $a_{-1} > 0$ are material constants. By Lemma 5.1.3, the energy due to volume change is never lower for $\boldsymbol{\varphi}$ than it is for $\boldsymbol{\varphi}^h$. However, with the aid of the following two propositions, we can show that the sum of both the volumetric energy and the Dirichlet energy may still be lower for $\boldsymbol{\varphi}$ than for $\boldsymbol{\varphi}^h$.

Proposition 5.1.4. *Let $\boldsymbol{\varphi}$ be given by* (5.1)*, and let $\boldsymbol{\varphi}^h$ be a pure homogeneous deformation given by* (5.4)*. Let the compressible neo-Hookean stored energy E be given by* (5.7)*, and let H be given by* (5.8)*. Define*

$$\Gamma(\lambda_1,\lambda_2,L) = b\lambda_1^2L^4 + \left[a\lambda_1^2 + c\lambda_2^2 + 2\alpha\left((\lambda_1\lambda_2)^2a_2 + \frac{a_{-1}}{\lambda_1\lambda_2}\right)\right]L^2 + d\lambda_2^2, \tag{5.9}$$

where a, b, c, and d are given by (5.5)*, and $\alpha := -\frac{15}{4} + 6\log(2)$. Then $E(\boldsymbol{\varphi}) < E(\boldsymbol{\varphi}^h)$ if and only if $\Gamma(\lambda_1,\lambda_2,L) < 0$.*

Proof. It may be shown by a direct calculation that

$$\int_\Omega H(\det(\nabla\boldsymbol{\varphi}(\mathbf{x})))\ \mathrm{d}\mathbf{x} = 2\alpha L\left((\lambda_1\lambda_2)^2a_2 + \frac{a_{-1}}{\lambda_1\lambda_2}\right).$$

Therefore, the total difference in compressible neo-Hookean stored energy is

$$
\begin{aligned}
\int_\Omega \frac{1}{2}|\nabla\boldsymbol{\varphi}(\mathbf{x})|^2 + &H(\det(\nabla\boldsymbol{\varphi}(\mathbf{x}))) - \frac{1}{2}|\nabla\boldsymbol{\varphi}^h(\mathbf{x})|^2 - H(\det(\nabla\boldsymbol{\varphi}^h(\mathbf{x})))\,\mathrm{d}\mathbf{x} \\
&= \lambda_1^2\left[\left(\frac{77}{24} - \frac{1}{9}\log(2)\right)L + \left(-\frac{59}{72} + 2\log(2)\right)L^3\right] + \frac{\lambda_2^2}{6}\left[-5L + \frac{4}{L}\right] \\
&\quad + 2\left(6\log(2) - \frac{15}{4}\right)L\left[(\lambda_1\lambda_2)^2 a_2 + \frac{a_{-1}}{\lambda_1\lambda_2}\right] \\
&= \frac{\lambda_1^2}{2}\left(aL + bL^3\right) + \frac{\lambda_2^2}{2}\left(cL + \frac{d}{L}\right) + 2\alpha L\left((\lambda_1\lambda_2)^2 a_2 + \frac{a_{-1}}{\lambda_1\lambda_2}\right).
\end{aligned}
$$

By our calculation, $\Gamma(\lambda_1, \lambda_2, L) = 2L(E[\boldsymbol{\varphi}] - E[\boldsymbol{\varphi}^h])$. So $E[\boldsymbol{\varphi}] < E[\boldsymbol{\varphi}^h]$ if and only if $\Gamma(\lambda_1, \lambda_2, L) < 0$. □

Remark 5.1.5. If we suppose that the top surface $\partial\Omega\cap((-2L, 2L)\times\{2\})$ of the material is stress-free when left undeformed, this implies that

$$
\begin{aligned}
\mathbf{0} &= \frac{\partial W(\mathbb{1})}{\partial \mathbf{F}}\mathbf{e}_2 \\
&= (1 + H'(1))\mathbf{e}_2.
\end{aligned}
$$

This holds if and only if $a_{-1} = 2a_2 + 1$. We will use a simple example where $a_2 = 1$ and $a_{-1} = 3$ for the following proposition.

Proposition 5.1.6. *Let $\boldsymbol{\varphi}$ be given by* (5.1)*, and let $\boldsymbol{\varphi}^h$ be a pure homogeneous deformation given by* (5.4)*. Let E be the compressible neo-Hookean stored energy given by* (5.7)*, and let H be given by* (5.8)*, with $a_2 = 1$ and $a_{-1} = 3$. Then for any pair $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1\lambda_2$ sufficiently small and such that the stress on $y = 2$ is zero:*

$$
\frac{\partial W(\nabla\boldsymbol{\varphi}^h)}{\partial \mathbf{F}}\mathbf{e}_2 = \mathbf{0}, \tag{5.10}
$$

there exists $L > 0$ such that $E[\boldsymbol{\varphi}] < E[\boldsymbol{\varphi}^h]$.

Proof. Note that (5.10) holds if and only if

$$
\frac{\lambda_2}{\lambda_1} + 2\lambda_1\lambda_2 - \frac{3}{(\lambda_1\lambda_2)^2} = 0,
$$

which, in particular, gives

$$
\lambda_2^2 = \frac{3}{\lambda_1\lambda_2} - 2(\lambda_1\lambda_2)^2, \tag{5.11a}
$$

$$
\lambda_1^{-2} = \frac{3}{(\lambda_1\lambda_2)^3} - 2. \tag{5.11b}
$$

By Proposition 5.1.4, $E(\boldsymbol{\varphi}) < E(\boldsymbol{\varphi}^h)$ if and only if $\Gamma(\lambda_1, \lambda_2, L) < 0$, where Γ is a quadratic function of L^2 given by (5.9). Since $b\lambda_1^2 > 0$ and $d\lambda_2^2 > 0$, the inequality $\Gamma(\lambda_1, \lambda_2, L) < 0$ is satisfied only if

$$a\lambda_1^2 + c\lambda_2^2 + 2\alpha\left((\lambda_1\lambda_2)^2 a_2 + \frac{a_{-1}}{\lambda_1\lambda_2}\right) < -2\lambda_1\lambda_2\sqrt{bd}, \tag{5.12}$$

For brevity, denote $\eta = \lambda_1\lambda_2$, and define

$$\begin{aligned} G(\eta) &:= \frac{a\eta^3}{3-2\eta^3} + 2(\alpha - c)\eta^2 + \frac{2\alpha + 3c}{\eta}, \\ \widetilde{\Gamma}(\eta, L) &:= \frac{bL^4\eta^3}{3-2\eta^3} + G(\eta)L^2 + \frac{d(3-2\eta^3)}{\eta}. \end{aligned} \tag{5.13}$$

By (5.9) and (5.11), $\Gamma(\lambda_1, \lambda_2, L) < 0$ if and only if $\widetilde{\Gamma}(\eta, L) < 0$, and (5.12) is equivalent to

$$G(\eta) + 2\sqrt{bd}\eta < 0. \tag{5.14}$$

There are exactly two solutions to the equation $G(\eta) + 2\sqrt{bd}\eta = 0$, say η_1 and η_2, where $\eta_1 < \sqrt[3]{\frac{3}{2}} < \eta_2$, see Figure 5-4. By (5.11), we must have $\eta < \sqrt[3]{\frac{3}{2}}$ (or else, e.g. $\lambda_2^2 < 0$), so (5.12) is satisfied if and only if $\eta < \eta_1$. It may be shown numerically that $\eta_1 \approx 0.757$.

The inequality $\lambda_1\lambda_2 < \eta_1$ together with (5.11) obtains

$$\lambda_2^2 > \frac{3}{\eta_1} - 2\eta_1^2 \approx 2.814, \tag{5.15}$$

$$\lambda_1^{-2} > \frac{3}{\eta_1^3} - 2 \approx 4.905. \tag{5.16}$$

Here we have a necessary condition for the existence of $\lambda_1, \lambda_2, L > 0$ such that (5.10) holds and $\Gamma(\lambda_1, \lambda_2, L) < 0$.

Now, given that (5.11) holds and $\eta < \eta_1 < \sqrt[3]{\frac{3}{2}}$, we have that $\widetilde{\Gamma}(\eta, L) < 0$ if and only if

$$\begin{aligned} 0 &> \eta(3-2\eta^3)\widetilde{\Gamma}(\eta, L) \\ &= b\eta^4 L^4 + \eta(3-2\eta^3)G(\eta)L^2 + d(3-2\eta^3)^2. \end{aligned}$$

So, given any pair λ_1 and λ_2 satisfying (5.15) and (5.16) respectively, there exists $L > 0$ such that

$$\frac{2b\eta^3 L^2}{3-2\eta^3} \in \left(-G(\eta) - \sqrt{G(\eta)^2 - 4bd\eta^2}, -G(\eta) + \sqrt{G(\eta)^2 - 4bd\eta^2}\right), \tag{5.17}$$

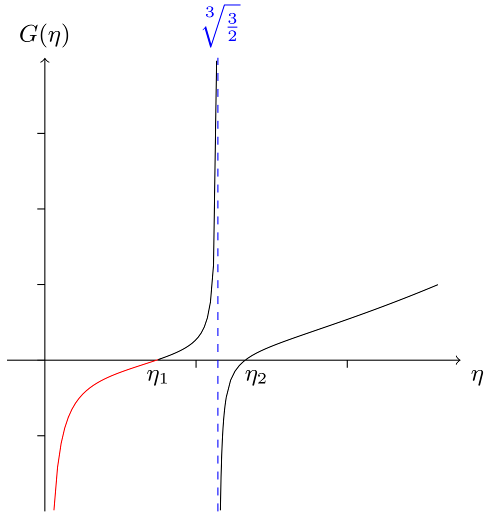


Figure 5-4: A plot of $G(\eta)$, where $\eta = \lambda_1\lambda_2$. The red segment, when $\eta < \eta_1$, is necessary for $\widetilde{\Gamma}(\eta, L) < 0$. The dashed blue asymptote is when $\eta = \sqrt[3]{\frac{3}{2}}$.

which is a nonempty interval since (5.14) holds. For these values of L, $\Gamma(\lambda_1, \lambda_2, L) < 0$. See Figure 5-5 for a plot of $\widetilde{\Gamma}(\eta, L)$. $\square$

Although precise conditions for favourable energy have been given in the above proof, the most important aspect of Proposition 5.1.6 is that a crease-like deformation $\boldsymbol{\varphi}$ given by (5.1), albeit not optimal for minimizing energy, is energetically more favourable than a comparable pure homogeneous deformation $\boldsymbol{\varphi}^h$, when λ_1, λ_2 satisfy (5.11) and L satisfies (5.17). Such principal stretches yield a compression ratio

$$\frac{\lambda_2}{\lambda_1} > \frac{3}{\eta_1^2} - 2\eta_1 \approx 3.714,$$

larger than Gent and Cho's experimental observation of 2.37 for creasing, and even Biot's estimate of 3.383 for wrinkling. This is almost certainly due to the fact that $\boldsymbol{\varphi}$ is not optimal in the sense of minimising stored energy.

An alternative perspective would be to first let $L > 0$, then find sufficient conditions for when λ_1 and λ_2 are such that $\Gamma(\lambda_1, \lambda_2, L) < 0$. Note that when η is small, by (5.13), we have that

$$\widetilde{\Gamma}(\eta, L) = \frac{3d + (2\alpha + 3c)L^2}{\eta} + o\left(\eta^{-1}\right).$$

So in this instance, a necessary condition for $\widetilde{\Gamma}(\eta, L) < 0$ is that

$$L > \sqrt{-\frac{3d}{2\alpha + 3c}} \approx 0.978, \tag{5.18}$$

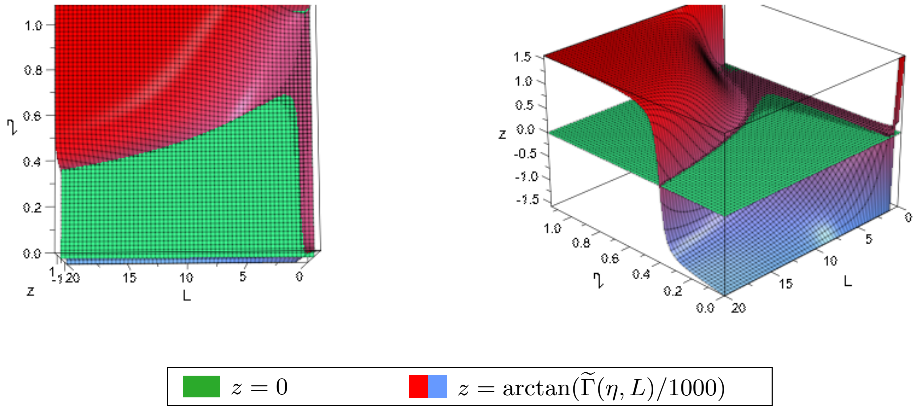


Figure 5-5: Two angles of a 3D plot of $z = \arctan(\widetilde{\Gamma}(\eta, L)/1000)$ (red), compared with 0 (green). We have divided $\widetilde{\Gamma}$ by 1000 and composed it with arctan for a smoother surface to more precisely illustrate when $\widetilde{\Gamma}(\eta) < 0$. Note that the peak of the green area in the leftmost figure occurs at $\eta = \eta_1$.

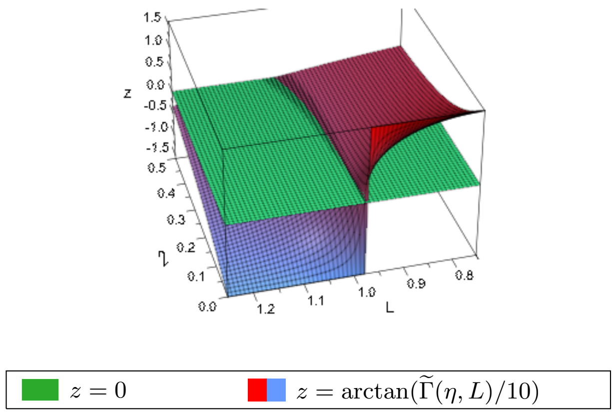


Figure 5-6: A 3D plot of $\arctan(\widetilde{\Gamma}(\eta, L)/10)$, now for $\eta \in (0, 0.5]$ and $L \in [0.75, 1.25]$, for a closer look at the behaviour as $\eta \to 0$ and $L \to 0.978$ (see (5.18)). Again, arctan has been used here for a smoother plot.

see Figure 5-6. To prove that for any L satisfying this bound, there exist λ_1, λ_2 such that $\Gamma(\lambda_1, \lambda_2, L) < 0$, is a tedious analytical task for which the author has not succeeded in completing.

For a more general case, it may be of interest to study an H of the form

$$H(d) = \sum_{j=-n}^{m} a_j d^j,$$

where $m, n \in \mathbb{N}$, and $a_j \in \mathbb{R}$.

A numerical study in Hong et al [HZS09] using an (incompressible) finite element analysis shows that for a very large block of an incompressible neo-Hookean material, a crease-like deformation has lower stored energy than that of a comparable pure homogeneous deformation when compressive strain is less than

$$\lambda_1 \approx 0.65, \tag{5.19}$$

consistent with experiments for creasing in rubber elastomers by Gent and Cho in [GC99]. This crease-like deformation is computed on a much finer mesh than the work presented in this section, and leads to a lower necessary compression for creasing to be energetically favourable than the present work. Moreover, each of these respective publications ([HZS09], [GC99]) note that this critical compression for creasing occurs earlier than Biot's prediction for surface wrinkling. Although failure of the complementing condition occurs at Biot's strain ratio, the results corresponding to instability at a compression ratio observed by Gent et al [GC99] (and computed by Hong et al [HZS09]) suggest that this occurs when quasiconvexity at the boundary fails. As a result, a crease may form before wrinkling modes, since the homogeneous state will be a weak local minimizer, but not a strong local minimiser.

5.2 Holomorphic deformations producing a crease

In this section, we build a crease-like deformation in two dimensions using holomorphic maps and the Joukowski transformation, and compare the energy of such a deformation with one that is purely homogeneous.

Consider the holomorphic complex functions $w_1 : H_- \to \mathbb{C}$, $w_2 : B_1(1+i) \to \mathbb{C}$,

$w_3 : U \to \mathbb{C}$ defined by

$$\begin{aligned}
w_1(z) &= 1 + i + \frac{i - \gamma z}{\gamma z - 1} = \frac{i\gamma z - 1}{\gamma z - 1}, \\
w_2(z) &= \frac{1}{2}\left(z + \frac{1}{z}\right), \\
w_3(z) &= ih\left(1 - \frac{2}{z}\right),
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \frac{i + e^{i\alpha}}{1 + e^{i\alpha}} && \alpha \in (-\pi, -\tfrac{\pi}{2}), \\
&= \frac{1+i}{2}\frac{1 + \cos(\alpha) + \sin(\alpha)}{1 + \cos(\alpha)} && \alpha \in (-\pi, -\tfrac{\pi}{2}), \\
&= -\frac{1+i}{2}\mu, && \mu \in \mathbb{R},\ \mu > 0,
\end{aligned}$$

h, the depth of the crease, is such that $\tan\left(\frac{\phi}{2}\right) = -\frac{1}{h}$, for some $\phi \in (-\pi, 0)$, $H_- = \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$, and $U \subset B_1(1)$ is a simple region bounded by the line parametrised in the following way

$$\begin{aligned}
w \in \partial U \iff \mathrm{Re}(w) &= \frac{1 + \cos(\theta)}{2}\left(1 + \frac{1}{3 + 2\cos(\theta) + 2\sin(\theta)}\right) \\
\mathrm{Im}(w) &= \frac{1 + \sin(\theta)}{2}\left(1 - \frac{1}{3 + 2\cos(\theta) + 2\sin(\theta)}\right).
\end{aligned}$$

These have the useful properties that

$$\begin{aligned}
w_1(H_-) &= B_1(1+i), \\
w_2(B_1(1+i)) &= U, \\
w_3(B_1(1)) &= H_-.
\end{aligned}$$

See Figures 5-7, 5-8, and 5-9. Define the map $w : H_- \to \mathbb{C}$ by the composition

$$\begin{aligned}
w(z) &= w_3(w_2(w_1(z))) \\
&= h\left(\mu z - \frac{i}{i\mu z + 1}\right),
\end{aligned} \tag{5.20}$$

which maps the half space H_- to the region illustrated in Figure 5-9. The real representation of this map is obtained by writing $z = x + iy$, $w = u + iv$ and expressing (u, v)

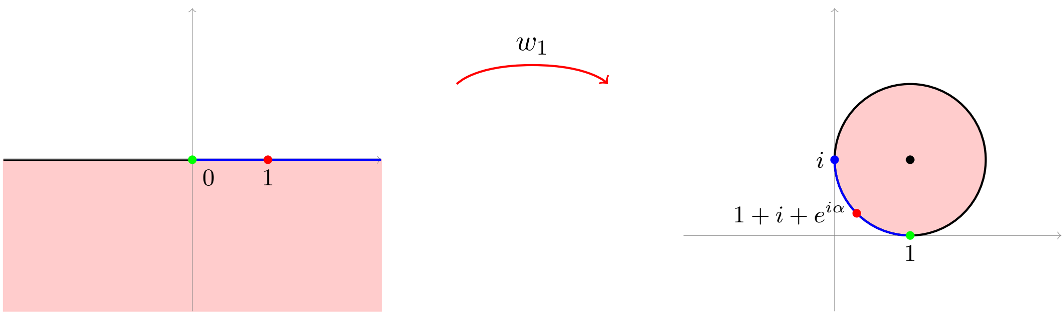


Figure 5-7: The map w_1 on H_-.

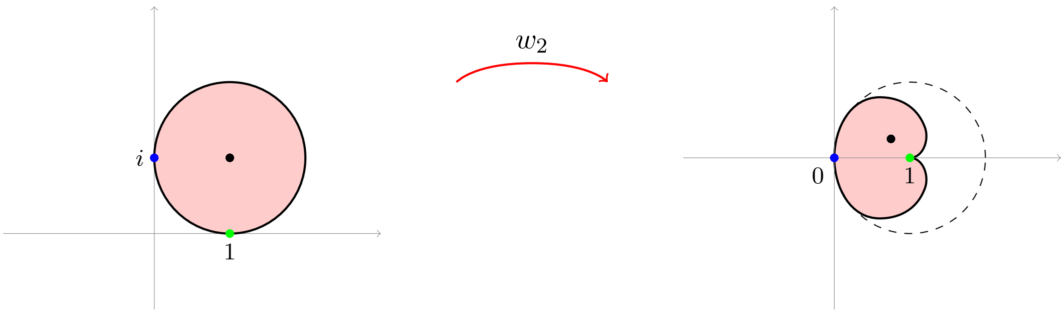


Figure 5-8: The Joukowski Transformation on $B_1(1+i)$.

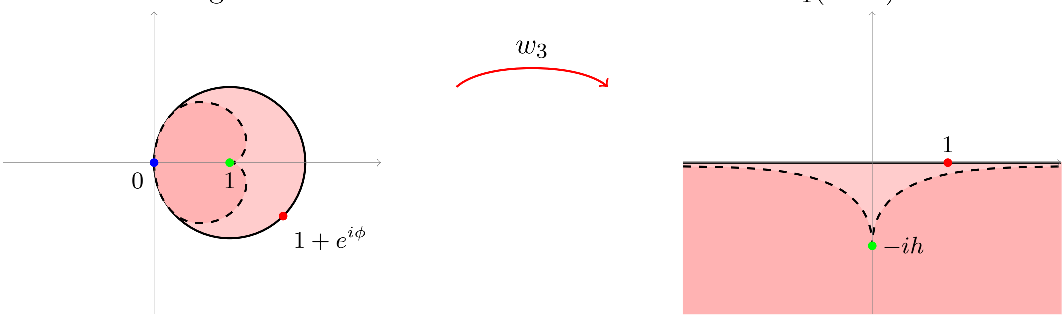


Figure 5-9: The map w_3 on U and on $B_1(1)$.

in terms of (x, y):

$$\left(\begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right) = h \left(\begin{array}{c} \mu x \left(1 - \frac{1}{\mu^2 x^2 + (1-\mu y)^2} \right) \\ \mu y - \frac{1-\mu y}{\mu^2 x^2 + (1-\mu y)^2} \end{array} \right).$$

Define

$$\boldsymbol{\varphi}^c(x,y) = \boldsymbol{\varphi}^h \left(\begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right) = \left(\begin{array}{c} \lambda_1 u(x,y) \\ \lambda_2 v(x,y) \end{array} \right), \tag{5.21}$$

where $\boldsymbol{\varphi}^h(x,y) = (\lambda_1 x, \lambda_2 y)^T$ is a pure homogeneous deformation. This deformation forms a cusp at $(x,y) = (0,0)$, and we wish to compare its energy to that of the comparable homogeneous deformation $\boldsymbol{\varphi}^h$. Firstly, this requires that $\boldsymbol{\varphi}^c \sim \boldsymbol{\varphi}^h$ for large $|\mathbf{x}|$, so

$$\mu = \frac{1}{h}.$$

To compare the stored energies of $\boldsymbol{\varphi}^c$ and $\boldsymbol{\varphi}^h$, note that since $w(z)$ is holomorphic, the Cauchy-Riemann equations hold, so

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1}{2}(u_x + i v_x) + \frac{1}{2i}(u_y + i v_y) = u_x + i v_x.$$

Hence

$$\left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^2 = \frac{1}{2} \left(u_x^2 + u_y^2 + v_x^2 + v_y^2 \right).$$

Furthermore,

$$\begin{aligned} |\nabla \boldsymbol{\varphi}^c|^2 &= \lambda_1^2 \left(u_x^2 + u_y^2 \right) + \lambda_2^2 \left(v_x^2 + v_y^2 \right) \\ &= \frac{\lambda_1^2 + \lambda_2^2}{2} \left(u_x^2 + u_y^2 + v_x^2 + v_y^2 \right) \\ &= \left(\lambda_1^2 + \lambda_2^2 \right) \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^2, \qquad (5.22) \\ \det(\nabla \boldsymbol{\varphi}^c) &= \lambda_1 \lambda_2 \left(u_x v_y - u_y v_x \right) \\ &= \lambda_1 \lambda_2 \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^2. \qquad (5.23) \end{aligned}$$

Hence, the behaviour of $\left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^2$ will determine the behaviour of the two principal invariants of $\sqrt{(\nabla\boldsymbol{\varphi}^c)^T\nabla\boldsymbol{\varphi}^c}$. Calculating this directly,

$$\begin{aligned}\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 &= h^2\left|\mu - \frac{\mu}{(i\mu z+1)^2}\right|^2 \\ &= \frac{|z|^2|2i-\mu z|^2}{|i\mu z+1|^4} \\ &= \mu^2\frac{(x^2+y^2)\left((\mu x)^2+(\mu y-2)^2\right)}{\left((\mu x)^2+(\mu y-1)^2\right)^2}. \end{aligned} \tag{5.24}$$

This is not integrable over the half space $H_- = \{(x,y)\in\mathbb{R}^2 \mid y<0\}$, since its far-field value is 1. Moreover, subtracting its far-field value of 1 yields an integrand that is not Lebegue integrable on $\mathbb{R}^2$, since it does not satisfy the hypotheses of Fubini's Theorem. Hence, calculating the difference in Dirichlet energy (such as in Subsection 5.1.1) for $\boldsymbol{\varphi}^c$ and $\boldsymbol{\varphi}^h$ cannot be carried out.

5.2.1 Dirichlet energy on a bounded domain

One possible way to proceed further with the use of (5.21) is through the use of proper Riemann integrals. Consider the stored energy over the bounded domain

$$\Omega = (-a,a)\times(-b,0), \tag{5.25}$$

where $a>0$ and $b>h$ are both taken to be very large. The downside of using this Ω is that $\boldsymbol{\varphi}$, given by (5.21), does not exactly match the boundary data on the sides $x=\pm a$ or the bottom $y=-b$, but the larger we take a and b, the smaller this discrepancy in the Dirichlet boundary data.

Lemma 5.2.1. *Let Ω be given by* (5.25)*, and let*

$$E_0(\boldsymbol{\varphi}) = \int_\Omega |\nabla\boldsymbol{\varphi}(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}. \tag{5.26}$$

Suppose $\frac{a}{b}=r>0$ is some fixed ratio of the dimensions of Ω. Then in the limit $a,b\to\infty$,

$$E_0(\boldsymbol{\varphi}^c) - E_0(\boldsymbol{\varphi}^h) \to \frac{\lambda_1^2+\lambda_2^2}{\mu^2}\left(-\frac{7\pi}{4}+4\arctan(r)\right).$$

Proof. By making the transformation $s = \mu x$ and $t = \mu y - 1$,

$$\begin{aligned}
&\int_{-b}^{0}\int_{-a}^{a} \mu^2 \frac{(x^2+y^2)\left((\mu x)^2 + (\mu y - 2)^2\right)}{\left((\mu x)^2 + (\mu y-1)^2\right)^2} - 1 \,\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{-\mu b-1}^{-1}\int_{-\mu a}^{\mu a} \frac{\left(s^2+(t+1)^2\right)\left(s^2+(t-1)^2\right)}{\left(s^2+t^2\right)^2} - 1 \,\mathrm{d}s\mathrm{d}t \\
&\quad = \frac{1}{\mu^2}\int_{1}^{\mu b+1}\int_{-\mu a}^{\mu a} \frac{2s^2+1-2t^2}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t \\
&\quad = \frac{1}{\mu^2}\left(\int_{1}^{\mu b+1}\int_{-\mu a}^{\mu a} \frac{1}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t + 4\arctan\left(\frac{\mu a}{\mu b+1}\right) - 4\arctan\left(\mu a\right)\right).
\end{aligned} \tag{5.27}$$

Note that

$$\int_{1}^{\mu b+1}\int_{-\mu a}^{\mu a} \frac{1}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t < \int_{1}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t = \frac{\pi}{4}, \tag{5.28}$$

and

$$\begin{aligned}
\frac{\pi}{4} &= \int_{1}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t \\
&= \int_{1}^{\mu b+1}\int_{-\mu a}^{\mu a} \frac{1}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t + \int_{\mu b+1}^{\infty}\left(\int_{-\infty}^{-\mu a} + \int_{\mu a}^{\infty}\right) \frac{1}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t \\
&< \int_{1}^{\mu b+1}\int_{-\mu a}^{\mu a} \frac{1}{\left(s^2+t^2\right)^2} \,\mathrm{d}s\mathrm{d}t + \frac{2}{((\mu a)^2 + (\mu b+1)^2)^2}.
\end{aligned} \tag{5.29}$$

Therefore, by (5.27), (5.28), and (5.29), we have the following upper and lower bounds for $\int_\Omega \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 - 1 \,\mathrm{d}\mathbf{x}$:

$$\begin{aligned}
&\int_{-b}^{0}\int_{-a}^{a} \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 - 1 \,\mathrm{d}x\mathrm{d}y < \frac{1}{\mu^2}\left(\frac{\pi}{4} + 4\arctan\left(\frac{\mu a}{\mu b+1}\right) - 4\arctan\left(\mu a\right)\right) \\
&\int_{-b}^{0}\int_{-a}^{a} \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 - 1 \,\mathrm{d}x\mathrm{d}y > \frac{1}{\mu^2}\left[\frac{\pi}{4} + 4\arctan\left(\frac{\mu a}{\mu b+1}\right) - 4\arctan\left(\mu a\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. - \frac{2}{((\mu a)^2 + (\mu b+1)^2)^2}\right].
\end{aligned}$$

For any fixed $\mu > 0$ and some fixed ratio $\frac{a}{b} = r > 0$, in the limit $a, b \to \infty$ we have that

$$\int_{-b}^{0}\int_{-a}^{a} \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 - 1 \,\mathrm{d}x\mathrm{d}y \to \frac{1}{\mu^2}\left(-\frac{7\pi}{4} + 4\arctan\left(r\right)\right). \tag{5.30}$$

Using (5.22) completes the proof. $\square$

Remark 5.2.2. Note that

$$-\frac{7\pi}{4} + 4\arctan(r) \in \left(-\frac{3\pi}{4}, \frac{\pi}{4}\right).$$

Hence, for sufficiently small r, and large enough a and b, $E_0(\boldsymbol{\varphi}^c) - E_0(\boldsymbol{\varphi}^h)$ can be made negative.

5.2.2 Compressible neo-Hookean stored energy on a bounded domain

Consider approximating the difference in compressible neo-Hookean stored energy

$$E[\boldsymbol{\varphi}] = \int_\Omega \frac{1}{2}|\nabla\boldsymbol{\varphi}(\mathbf{x})|^2 + H(\det(\nabla\boldsymbol{\varphi}(\mathbf{x})))\,\mathrm{d}\mathbf{x}, \tag{5.31}$$

between $\boldsymbol{\varphi}^c$ and $\boldsymbol{\varphi}^h$ on a bounded rdomain Ω. A problem arises in this case from a natural requirement on H that $H(d) \to \infty$ as $d \to 0$. In fact, by (5.23) and (5.24), any class of H with $H(d) \sim \frac{1}{\sqrt{d}}$ (or faster) as $d \to 0$ will imply that the corresponding $H(\det\nabla\boldsymbol{\varphi}^c) - H(\det\nabla\boldsymbol{\varphi}^h)$ is not integrable.

In hindsight, holomorphic maps may be unsuitable for constructing reasonable competing deformations producing a crease. For incompressible elasticity, Theorem 4.2.6, [Bev14], and [Cia18] suggest that the double-covering map is at least a close approximation of the local behaviour of a crease-like deformation where a sulcus forms[3]. A complex representation of the double-covering map is

$$w_{DC}(z) = \frac{1}{\sqrt{2}|z|}\left(x^2 - y^2 + 2ixy\right) = \frac{z^2}{\sqrt{2}|z|}, \tag{5.32}$$

but the appearance of $|z|$, and consequently $\bar{z}$, is not consistent with the use of analytic maps local to wherever the sulcus forms. In fact, for small z, by expanding w given by (5.20) using a Lorentz series, we have

$$w(z) = -ih + \frac{iz^2}{h} + \dots, \tag{5.33}$$

which yields an infinitesimally small determinant of the deformation gradient for small z by (5.23). Creasing experimentally seems to occur with finite and nonzero principal invariants local to the crease, so $\boldsymbol{\varphi}^c$, by itself, is an inappropriate approximation close to the crease.

[3] Also remarked by Silling in [Sil91].

Noting the asymptotic behaviour of (5.32) and (5.33) as $z \to 0$, one possible approach is to try to match the quadratic terms in z in (a multiple $\alpha > 0$ of) (5.32) and (5.33) on a circle $|z| = \frac{\alpha h}{\sqrt{2}}$. An example of a deformation satisfying this condition is one given by

$$\boldsymbol{\varphi}(\mathbf{x}) = \begin{cases} \boldsymbol{\varphi}^c(\mathbf{x}) & \text{for } |\mathbf{x}| > \frac{\alpha h}{\sqrt{2}}, \\ \operatorname{diag}(\lambda_1, \lambda_2)\mathbf{Q}\left(\frac{\pi}{2}\right)\boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x}) & \text{for } |\mathbf{x}| \leq \frac{\alpha h}{\sqrt{2}}. \end{cases}$$

If $\alpha h \ll 1$, $\boldsymbol{\varphi}$ is approximately continuous across $|\mathbf{x}| = \frac{\alpha h}{\sqrt{2}}$. Moreover,

$$\det \nabla\boldsymbol{\varphi} \geq \lambda_1\lambda_2 \max\left\{\alpha^2, \frac{\alpha^2}{2}\frac{\frac{\alpha^2}{2}+4}{\left(\frac{\alpha}{\sqrt{2}}+1\right)^4}\right\}$$

is bounded away from zero. We then have that $H(\det\nabla\boldsymbol{\varphi}) - H(\det\nabla\boldsymbol{\varphi}^h)$ is integrable, so (5.31) will be finite. Unfortunately, finding the total compressible neo-Hookean stored energy of this $\boldsymbol{\varphi}$ has proved too algebraically cumbersome to complete. Ciarletta [Cia18] uses some form of asymptotic matching to a double-covering map and from this claims that the incompressible neo-Hookean stored energy of an $8L \times L$ block in its reference state undergoing a crease-like deformation is lower than that of a comparable pure homogeneous deformation when the stretch λ_1 reaches a critical value of 0.6372 (see [Cia18, Figure 4]). Unfortunately, the complete details of this calculation are unclear and not given in [Cia18] (see also the peer reviewers' comments on the paper found on https://doi.org/10.1038/s41467-018-02979-6).

Chapter 6

Concluding remarks

6.1 The second variation with a slip boundary condition

In Definition 1.2.11, we define the second variation of E (given by (1.20)) via linear variations $\boldsymbol{\varphi} + \epsilon\mathbf{u}$. This is a special case of a more rigorous approach, where we let $\boldsymbol{\varphi}_0 \in \mathcal{A}$ be a solution to the boundary value problem (1.16), (1.21), and (1.22), and let $\boldsymbol{\varphi} : \Omega \times (-\delta, \delta) \to \mathbb{R}^n$ be a family of admissible deformations satisfying (1.21) and such that $\boldsymbol{\varphi}(\cdot, 0) = \boldsymbol{\varphi}_0$. The second variation is defined by

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} E[\boldsymbol{\varphi}(\cdot,\epsilon)]\bigg|_{\epsilon=0} &= \int_\Omega \mathbf{C}(\nabla\boldsymbol{\varphi}_0(\mathbf{x}))[\dot{\boldsymbol{\varphi}}(\mathbf{x},0), \dot{\boldsymbol{\varphi}}(\mathbf{x},0)] + \widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}_0(\mathbf{x})) \cdot \nabla\ddot{\boldsymbol{\varphi}}(\mathbf{x},0)\ \mathrm{d}\mathbf{x} \\ &\qquad - \int_{\partial\Omega_T} \ddot{\boldsymbol{\varphi}}(\mathbf{x},0) \cdot \mathbf{t}(\mathbf{x})\ \mathrm{d}S(\mathbf{x}) \\ &= \int_\Omega \mathbf{C}(\nabla\boldsymbol{\varphi}_0(\mathbf{x}))[\dot{\boldsymbol{\varphi}}(\mathbf{x},0), \dot{\boldsymbol{\varphi}}(\mathbf{x},0)]\ \mathrm{d}\mathbf{x} \\ &\qquad + \int_{\partial\Omega} \ddot{\boldsymbol{\varphi}}(\mathbf{x},0) \cdot \Big(\widehat{\mathbf{S}}(\nabla\boldsymbol{\varphi}(\mathbf{x}))\mathbf{n}(\mathbf{x}) - \mathbf{t}(\mathbf{x})\Big)\ \mathrm{d}S(\mathbf{x}). \end{aligned} \tag{6.1}$$

In Subsection 1.2.6, where we have the case $\partial\Omega = \partial\Omega_D \cup \partial\Omega_T$, we have by (1.21) that $\dot{\boldsymbol{\varphi}}(\mathbf{x},0) = \ddot{\boldsymbol{\varphi}}(\mathbf{x},0) = 0$ for $\mathbf{x} \in \partial\Omega_D$. Therefore, the boundary integral on the second line of (6.1) is zero. Hence, if we let $\mathbf{u} = \dot{\boldsymbol{\varphi}}(\cdot,0)$, we could instead consider the family of deformations $\boldsymbol{\varphi}(\cdot,\epsilon) = \boldsymbol{\varphi}_0 + \epsilon\mathbf{u}$ for simplicity.

In the more general case $\partial\Omega = \partial\Omega_D \cup \partial\Omega_T \cup \partial\Omega_S$ where $\partial\Omega_S$ is nonempty, we have that admissible deformations $\boldsymbol{\varphi}(\cdot,\epsilon)$ satisfy the constraint $\boldsymbol{\varphi}(\mathbf{x},\epsilon) \in \partial Y$ for $\mathbf{x} \in \partial\Omega_S$. Therefore, $\dot{\boldsymbol{\varphi}}(\mathbf{x},0) \cdot \mathbf{N}(\boldsymbol{\varphi}_0(\mathbf{x})) = 0$ for $\mathbf{x} \in \partial\Omega_S$, and $\ddot{\boldsymbol{\varphi}}(\cdot,0)$ depends on the curvature of ∂Y. In this case, the second variation cannot be simplified in a similar way to the case $\partial\Omega_S = \varnothing$, since the integral over $\partial\Omega$ in (6.1) is not necessarily zero. This term does not appear in the theory considered in Subsection 1.2.6. It would be of interest to extend the work of [SS87] and [SS89] on the second variation to include slip boundary conditions.

6.2 The incompressible double-covering map as a global energy minimiser in three dimensions

In Theorem 4.2.8, we proved that the incompressible double-covering map $\widetilde{\boldsymbol{\varphi}}_{DC\gamma}$ is the global minimiser of the stored energy E^{inc} over the set of admissible deformations $\widetilde{\mathcal{A}}$, where $\widetilde{\boldsymbol{\varphi}}_{DC\gamma}$ is given by

$$\widetilde{\boldsymbol{\varphi}}_{DC\gamma} = \begin{pmatrix} \frac{1}{\sqrt{2\gamma(x_1^2+x_2^2)}}(x_1^2 - x_2^2) \\ \frac{1}{\sqrt{2\gamma(x_1^2+x_2^2)}}2x_1x_2 \\ \gamma x_3 \end{pmatrix}, \tag{6.2}$$

the stored energy E^{inc} being given by

$$E^{\text{inc}}[\boldsymbol{\varphi}] = \int_\Omega h^{\text{inc}}(|\nabla\boldsymbol{\varphi}(\mathbf{x})|, |\text{Cof}\nabla\boldsymbol{\varphi}(\mathbf{x})|)\, \mathrm{d}\mathbf{x},$$

for some h^{inc} convex and increasing in each argument, and

$$\widetilde{\mathcal{A}} = \{\boldsymbol{\varphi} \in C^1(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3) \mid \det(\nabla\boldsymbol{\varphi}) = 1,\ \boldsymbol{\varphi} \text{ satisfies (4.67), (4.68), (4.69),} \\ \text{and } \boldsymbol{\varphi} \text{ is of the form (4.70)}\}.$$

Note that $\widetilde{\mathcal{A}}$ only allows deformations of the form

$$\boldsymbol{\varphi}(\mathbf{x}) = \begin{pmatrix} \varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \\ \gamma x_3 \end{pmatrix}.$$

One possible way to extend this result to include more general deformations is to first prove that the global minimiser is in $\widetilde{\mathcal{A}}$, and then to simply apply Theorem 4.2.8. A promising method to achieve this first step is via the minimising properties of isochoric plane-strain deformations studied in [SS10] (see in particular [SS10, Lemma 2.4]).

6.3 Predicting Gent and Cho's estimate via loss of quasiconvexity at the boundary for incompressible hyperelasticity

A key observation in understanding the difference between surface creasing and surface wrinkling is that surface wrinkling is a type of instability resulting in loss of weak local minimality (see Definition 1.2.6). The loss of strong, but not weak, local minimality allows variations of class C^0, permitting crease-like behaviour. To verify that creasing occurs, it is sufficient to show that quasiconvexity at the boundary is lost at a compression with ratio presumably close to that which Gent and Cho [GC99] have observed (or at least at a compression before Biot's [Bio63] prediction).

Definition 6.3.1. Let $D_{\mathbf{N}}$ be a standard boundary domain with normal $\mathbf{N}$ and interior $\Gamma_{\mathbf{N}}$ of $\partial D_{\mathbf{N}} \cap \partial H_{\mathbf{N}}$ (see Definition 1.2.20). The function W^{inc} is said to be $W^{1,\infty}$-*quasiconvex at the boundary* at $(\mathbf{F}_0, \mathbf{N})$ if there exists a constant vector $\mathbf{t}_0 \in \mathbb{R}^n$ such that

$$\int_{D_{\mathbf{N}}} W^{\text{inc}}(\nabla\boldsymbol{\xi}(\mathbf{y})\mathbf{F}_0) - W^{\text{inc}}(\mathbf{F}_0)\ \mathrm{d}\mathbf{y} - \int_{\Gamma_{\mathbf{N}}} \mathbf{t}_0 \cdot (\boldsymbol{\xi}(\mathbf{y}) - \mathbf{y})\ \ \mathrm{d}S(\mathbf{y}) \geq 0 \tag{6.3}$$

for all isochoric $\boldsymbol{\xi} \in W^{1,\infty}(D_{\mathbf{N}}, \mathbb{R}^n)$ (compare with Definition 1.2.31) such that $\boldsymbol{\xi}(\mathbf{y}) = \mathbf{y}$ for $\mathbf{y} = \partial D_{\mathbf{N}} \setminus \Gamma_{\mathbf{N}}$ (in the sense of trace).

We are interested in whether the (incompressible) pure homogeneous deformation $\boldsymbol{\varphi}^h$, given by

$$\boldsymbol{\varphi}^h(\mathbf{x}) = \mathbf{D}\mathbf{x}, \tag{6.4}$$

is a strong local minimiser for some system, where

$$\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

We consider the standard boundary domain in two dimensions given by

$$D_{\mathbf{N}} = \{\mathbf{y} \in \mathbb{R}^2 \mid |\mathbf{y}|^2 < 1,\ y_2 < 0\}, \tag{6.5}$$

with corresponding normal $\mathbf{N} = \mathbf{e}_2$ and $\Gamma_{\mathbf{N}} = (-1, 1) \times \{0\}$ (see Definition 1.2.20). Let us say that W^{inc} is $W^{1,\infty}$-quasiconvex at the boundary at $(\mathbf{D}, \mathbf{N})$, and with the choice $\mathbf{t}_0 = 0$, so that

$$\int_{D_{\mathbf{N}}} W^{\text{inc}}(\nabla\boldsymbol{\xi}(\mathbf{y})\mathbf{D}) - W^{\text{inc}}(\mathbf{D})\ \mathrm{d}\mathbf{y} \geq 0 \tag{6.6}$$

for all isochoric $\boldsymbol{\xi} \in W^{1,\infty}(D_{\mathbf{N}}, \mathbb{R}^2)$ with $\boldsymbol{\xi}(\mathbf{y}) = \mathbf{y}$ on $\partial D_{\mathbf{N}} \setminus \Gamma_{\mathbf{N}}$ (in the sense of trace).

Remark 6.3.2. It is at present an open question as to whether Theorem 1.2.33 in [Mac05] holds under the hypothesis $\boldsymbol{\xi} \in W^{1,\infty}(D_{\mathbf{N}}, \mathbb{R}^n)$ (see also Remark 1.2.25 for the analogue in the compressible case).

We now consider the example stored energy function

$$W^{\mathrm{inc}}(\mathbf{F}) = \frac{1}{2}|\mathbf{F}|^2, \quad \mathbf{F} \in M_1^{2\times 2}.$$

We suspect that at some critical compression, quasiconvexity at the boundary is lost, with the material preferring to develop a small crease instead of remaining homogeneous. Therefore, we propose that for some $\delta \in (0,1)$, such a loss of quasiconvexity at the boundary occurs when ξ takes the form

$$\boldsymbol{\xi}(\mathbf{y}) = \begin{cases} \boldsymbol{\pi}(\mathbf{y}), & \mathbf{x} \in D_{\mathbf{n}} \cap B_\delta(0), \\ \mathbf{u}(\mathbf{y}), & \text{otherwise}, \end{cases} \tag{6.7}$$

where

$$\boldsymbol{\pi}(\mathbf{y}) = \mathbf{Q}\left(\tfrac{\pi}{2}\right) \boldsymbol{\varphi}_{DC_1}(\mathbf{y}) + h\mathbf{e}_2,$$

$\boldsymbol{\varphi}_{DC_1}$ is the double-covering map given by (4.42) with $\alpha = 1$, $\mathbf{Q}(\frac{\pi}{2})$ is given by

$$\mathbf{Q}\left(\tfrac{\pi}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$h \in (-1, 0)$ is some constant, and $\mathbf{u} \in C^2(D_{\mathbf{N}} \setminus B_\delta(0), \mathbb{R}^2)$ is some function with $\mathbf{u}(\mathbf{y}) = \mathbf{y}$ on $\partial D_{\mathbf{N}} \setminus \Gamma_{\mathbf{N}}$. Explicitly, $\boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi}(\mathbf{y}) = \begin{pmatrix} \frac{-2y_1y_2}{\sqrt{2}|\mathbf{y}|} \\ \frac{y_1^2 - y_2^2}{\sqrt{2}|\mathbf{y}|} + h \end{pmatrix}.$$

To find the optimal h and $\mathbf{u}$, we suppose $\mathbf{u}$ is a local minimiser of the (constrained) functional

$$Q[\mathbf{u}] = \int_{D_{\mathbf{N}} \setminus B_\delta(0)} |\nabla \mathbf{u}(\mathbf{y})\mathbf{D}|^2 - |\mathbf{D}|^2 \, \mathrm{d}\mathbf{y}, \tag{6.8}$$

over the set of admissible functions

$$\mathcal{A} = \{\mathbf{u} : D_{\mathbf{N}} \setminus B_\delta(0) \to \mathbb{R}^2 \mid \det \nabla \mathbf{u} = 1, \ \mathbf{u}(\mathbf{y}) = \mathbf{y} \text{ on } D_{\mathbf{N}} \setminus \Gamma_{\mathbf{N}}, \\ \mathbf{u}(\mathbf{y}) = \boldsymbol{\pi}(\mathbf{y}) \text{ on } \partial B_\delta(0)\}.$$

The Euler-Lagrange equations associated to this problem are given by

$$\frac{\partial}{\partial y_\alpha}\left[\frac{\partial u_i}{\partial y_\beta}(\mathbf{D}^2)_{\beta\alpha} - p(\mathbf{y})\mathrm{Cof}(\nabla\mathbf{u})_{i\alpha}\right] = 0, \qquad \mathbf{y}\in D_{\mathbf{N}}\setminus B_\delta(0), \tag{6.9a}$$

$$\det(\nabla\mathbf{u}(\mathbf{y})) = 1, \qquad \mathbf{y}\in D_{\mathbf{N}}\setminus B_\delta(0), \tag{6.9b}$$

$$\mathbf{u}(\mathbf{y}) = \mathbf{y}, \qquad \mathbf{y}\in \partial D_{\mathbf{N}}\setminus \Gamma_{\mathbf{n}}, \tag{6.9c}$$

$$\mathbf{u}(\mathbf{y}) = \boldsymbol{\pi}(\mathbf{y}), \qquad \mathbf{y}\in D_{\mathbf{N}}\cap \partial B_\delta(0), \tag{6.9d}$$

$$\left(\frac{\partial u_i}{\partial y_\beta}(\mathbf{D}^2)_{\beta\alpha} - p(\mathbf{y})\mathrm{Cof}(\nabla\mathbf{u})_{i\alpha}\right)N_\alpha = 0, \qquad \mathbf{y}\in \Gamma_{\mathbf{N}}, \tag{6.9e}$$

see Figure 6-1.

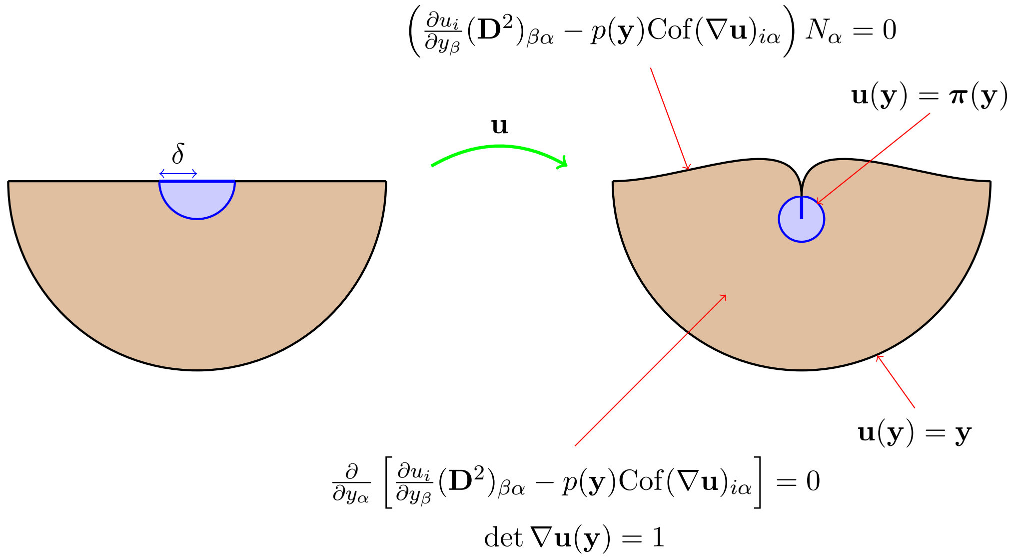


Figure 6-1

It is helpful to express (6.9a) component-wise

$$\lambda_1^2\frac{\partial^2 u_1}{\partial y_1^2} + \lambda_2^2\frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial p}{\partial y_1}\frac{\partial u_2}{\partial y_2} + \frac{\partial p}{\partial y_2}\frac{\partial u_2}{\partial y_1} = 0, \tag{6.10a}$$

$$\lambda_1^2\frac{\partial^2 u_2}{\partial y_1^2} + \lambda_2^2\frac{\partial^2 u_2}{\partial y_2^2} + \frac{\partial p}{\partial y_1}\frac{\partial u_1}{\partial y_2} - \frac{\partial p}{\partial y_2}\frac{\partial u_1}{\partial y_1} = 0. \tag{6.10b}$$

Note that if u_1 and u_2 are both solutions of the Laplace eigenvalue problem (with no sum on repeated indices)

$$\triangle_\lambda u_i = v_i u_i, \tag{6.11}$$

where

$$\triangle_\lambda u_i = \lambda_1^2\frac{\partial^2 u_i}{\partial y_1^2} + \lambda_2^2\frac{\partial^2 u_i}{\partial y_2^2},$$

such that (6.9b) holds, with the choice

$$p(\mathbf{y}) = \frac{1}{2}\left(v_1 u_1(\mathbf{y})^2 + v_2 u_2(\mathbf{y})^2\right), \tag{6.12}$$

then $\mathbf{u}$ satisfies (6.10), for *any* pair of eigenvalues $v_1, v_2 \in \mathbb{R}$.

We conjecture that in the limit $\delta \to 0$, there exists a solution of (6.9) such that (6.6) is violated for some $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2)$, and $h \in (-1, 0)$. Furthermore, the corresponding λ_1 in which (6.6) is violated is conjectured to be strictly greater than $\sqrt{\xi^*} \approx 0.544$, where $\xi^* \approx 0.296$ is the only real root of the cubic polynomial F, given by (3.73) (corresponding to loss of weak local minimality, see (3.62)). That is, a loss of W^∞-quasiconvexity at the boundary occurs at a lower compression than Biot's prediction for wrinkling [Bio63]. We believe this method will give an accurate estimation for creasing in the limit $\delta \to 0$. We have essentially desingularised the problem by asserting that W^∞-quasiconvexity at the boundary is lost via a crease-like variation which looks like the double-covering map at the only singularity, proposed to be at the origin. Moreover, the variation in question is assumed to be C^2 away from the crease, hence appropriately found by the Euler-Lagrange equations for Q. Note that if such a solution exists and (6.6) is violated, this is only a sufficient condition for the loss of W^∞-quasiconvexity at the boundary. The double-covering map $\boldsymbol{\varphi}_{DC_1}$ is chosen as the probable candidate for this work mainly due to its minimising properties discussed in Section 4.2, and Ciarletta's work in [Cia18]. However, there appear to be no experimental studies confirming or contradicting that this behaviour occurs in the creasing of rubber elastomers. Nevertheless, it is a promising topic for future research.

6.4 Modelling crease formation for a rubber diaphragm

As outlined in the introduction, this project was initially motivated by the problem encountered by Weatherford International, where creases would form on the inner surface of a synthetic oil-filled rubber diaphragm in the shape of a slightly tapered tube that is used in their steerable drilling system (see Figure 1-2 and Figure 1-3). To model this problem, since the creases form parallel to the axis of symmetry, we consider a two-dimensional cross-section.

Considering the two dimensional problem in which a (compressible) rubber material occupies the annulus in $\mathbb{R}^2$, given by

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid B < |\mathbf{x}| < A\}.$$

At large pressures and temperatures experienced underground (approximately 1800 bar and 180°C), the annulus initially expands. Over time with repeated use, the rubber annulus undergoes "set",[1] whereby its natural configuration becomes the expanded annulus. When the drilling system returns to the surface, the pressure difference causes sufficient compression for buckling to occur, with 4 observed buckling folds (see Figure 1-3). Creases are observed to form on the inner surface at each fold, where the post buckling compression is largest (see Figure 6-2).

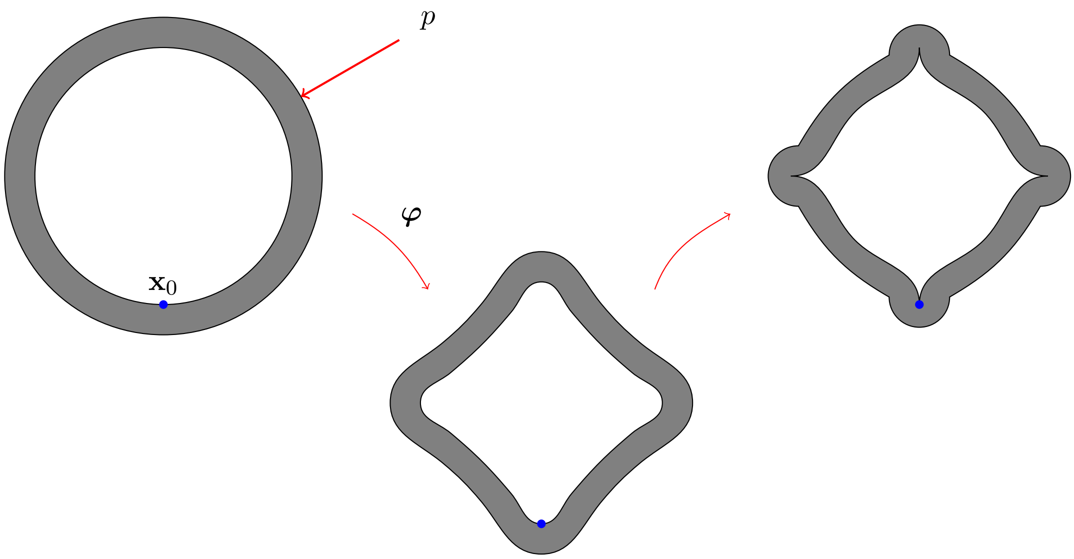


Figure 6-2

Let $\boldsymbol{\varphi}$ be an equilibrium deformation corresponding to the initial (four fold) buckling prior to the formation of a crease. Suppose $\mathbf{x}_0 \in \partial\Omega$ lies on the inner surface of the annulus where a buckle fold occurs and is such that the compressive strain at $\mathbf{x}_0$ is lower than at any other point on fold. Let E be the stored energy of the material, and let W be its corresponding stored energy function. We have that

(i) by Theorem 1.2.24, a necessary condition for $\boldsymbol{\varphi}$ to be a strong local minimiser of E is that for all $\mathbf{x} \in \partial\Omega$ with normal $\mathbf{n}$, W is quasiconvex at the boundary at $(\nabla\boldsymbol{\varphi}(\mathbf{x}), \mathbf{n})$ (see Definition 1.2.21 and Remark 1.2.25);

(ii) by Theorem 1.2.26 and [SS87, Proposition 4.2], a necessary condition for $\boldsymbol{\varphi}$ to be a weak local minimiser of E is that for all $\mathbf{x} \in \partial\Omega$, the elasticity tensor $\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x}))$ (defined by (2.6)) satisfies the Legendre-Hadamard condition 1.29, the pair $(\mathbf{C}(\nabla\boldsymbol{\varphi}(\mathbf{x})), \mathbf{n})$ satisfies Agmon's condition, and the supplementary condition Theorem 1.2.26 (3) holds.

[1]Set can occur in elastomers and polymers for a number of reasons. A discussion of these is beyond the scope of this thesis.

We conjecture that the bucked state $\boldsymbol{\varphi}$ is a weak local minimiser, and that a crease forms at $\mathbf{x}_0$ since it is locally the first boundary point experiencing sufficient compression for quasiconvexity at the boundary to fail.

Many rubbers are modelled as an incompressible material. A similar explanation for crease formation can be proposed in the case of an incompressible stored energy E^{inc} and corresponding stored energy function W^{inc}. In this case,

(i) we conjecture that a necessary condition for $\boldsymbol{\varphi}$ to be a strong local minimiser of E^{inc} is that for all $\mathbf{x} \in \partial\Omega$ with normal $\mathbf{N}$, W^{inc} is $W^{1,\infty}$-quasiconvex at the boundary at $(\nabla\boldsymbol{\varphi}(\mathbf{x}), \mathbf{N})$ (see Remark 6.3.2);

(ii) by Theorem 3.1.5 and Theorem 1.2.38, a necessary condition for $\boldsymbol{\varphi}$ to be a weak local minimiser of E^{inc} is that for all $\mathbf{x}_0 \in \partial\Omega$, the tensor $\mathbb{K}$ defined by (1.50) satisfies the Legendre-Hadamard condition (1.54), Agmon's condition (Theorem 1.2.38 (2)), and the supplementary condition Theorem 1.2.38 (3).

In this incompressible case we conjecture that the buckled state $\boldsymbol{\varphi}$ is a weak local minimiser and that a crease forms at a point $\mathbf{x}_0 \in \partial\Omega$ at which $W^{1,\infty}$-quasiconvexity at the boundary fails.

Appendix A

Appendix

A.1 List of Symbols

For convenience of the reader, we list below our notation convention as introduced by chapter:

A.1.1 Chapter 1

$M_+^{n\times n}$	the set of $n \times n$ matrices with positive determinant
$M_1^{n\times n}$	the set of $n \times n$ matrices with determinant equal to 1
$SO(n)$	the set of $n \times n$ orthogonal matrices with determinant 1
$\mathbf{X}^T$	the transpose of a vector or matrix $\mathbf{X}$
$\mathbf{u} \otimes \mathbf{v}$	the tensor product of two vectors $\mathbf{u}$ and $\mathbf{v}$, given component-wise by $u_i v_j$
$\mathbf{A}^{-1}$	the inverse of a nonsingular matrix $\mathbf{A}$
$\mathbb{1}$	the unit matrix, with (i,j)th component δ_{ij}
$\det(\mathbf{A})$	the determinant of a matrix $\mathbf{A}$
$\mathrm{Cof}(\mathbf{A})$	the cofactor matrix of a matrix $\mathbf{A}$, given component-wise by $(-1)^{i+j}M_{ij}(\mathbf{A})$, where $M_{ij}(\mathbf{A})$ is the (i,j)th minor of $\mathbf{A}$
$\mathrm{tr}(\mathbf{A})$	the trace of a matrix $\mathbf{A}$
$\sqrt{\mathbf{G}}$	the square root of a symmetric positive definite matrix $\mathbf{G}$, given by the unique symmetric positive definite matrix $\mathbf{U}$ such that $\mathbf{U}^2 = \mathbf{G}$
$\mathbf{A} \cdot \mathbf{B}$	the matrix product between matrices $\mathbf{A}$ and $\mathbf{B}$, given by $\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = A_{ij}B_{ij}$

$\mathrm{diag}(\lambda_1,\dots,\lambda_n)$	The $n\times n$ diagonal matrix with $(\lambda_1,\dots,\lambda_n)$ on the leading diagonal
$\mathbf{C}[\mathbf{A}]$	a four-tensor $\mathbf{C}$ acting on a matrix $\mathbf{A}$, given component-wise by $C^{ij}_{\alpha\beta}A_{j\beta}$
$\mathbf{C}[\mathbf{A},\mathbf{B}]$	a four-tensor product on matrices $\mathbf{A}$ and $\mathbf{B}$, given by $C^{ij}_{\alpha\beta}A_{i\alpha}B_{j\beta}$.

A.1.2 Chapter 2

$\mathbf{D}$	The diagonal $n\times n$ matrix with λ_i in the (i,i) entry, for $i=1,\dots,n$ and 0 elsewhere
Φ_i	$\Phi_{,i}(\lambda_1,\dots,\lambda_n)$, for $i,j=1,\dots,n$
Φ_{ij}	$\Phi_{,ij}(\lambda_1,\dots,\lambda_n)$, for $i,j=1,\dots,n$
Ψ_{ij}	$\frac{\lambda_i\Phi_i-\lambda_j\Phi_j}{\lambda_i^2-\lambda_j^2}$, for $i,j=1,\dots,n$
Θ_{ij}	$\frac{\lambda_j\Phi_i-\lambda_i\Phi_j}{\lambda_i^2-\lambda_j^2}$, for $i,j=1,\dots,n$
$\mathbf{C}$	The elasticity tensor evaluated at $\mathbf{F}=\mathbf{D}$, given by $\frac{\partial^2 W(\mathbf{D})}{\partial\mathbf{F}^2}$

A.1.3 Chapter 3

$\mathcal{T}(\mathbf{F})$	The set of matrices $\mathbf{G}\in M^{n\times n}$ satisfying $\mathrm{Cof}\mathbf{F}\cdot\mathbf{G}=0$
$\mathcal{N}(\mathbf{F},\mathbf{G})$	The set of matrices $\mathbf{K}\in M^{n\times n}$ satisfying $\mathrm{Cof}'(\mathbf{F})[\mathbf{G},\mathbf{G}]+\mathrm{Cof}\mathbf{D}\cdot\mathbf{K}=0$
$\mathbf{D}$	The diagonal $n\times n$ matrix with λ_i in the (i,i) entry, for $i=1,\dots,n$ and 0 elsewhere
Φ_i	$\Phi_{,i}(\lambda_1,\dots,\lambda_n)$, for $i,j=1,\dots,n$
Φ_{ij}	$\Phi_{,ij}(\lambda_1,\dots,\lambda_n)$, for $i,j=1,\dots,n$
Ψ_{ij}	$\frac{\lambda_i\Phi_i-\lambda_j\Phi_j}{\lambda_i^2-\lambda_j^2}$, for $i,j=1,\dots,n$
Θ_{ij}	$\frac{\lambda_j\Phi_i-\lambda_i\Phi_j}{\lambda_i^2-\lambda_j^2}$, for $i,j=1,\dots,n$
Λ_n	The set of vectors $(v_1,\dots,v_n)\in(0,\infty)^n$ satisfying $v_1\dots v_n=1$
$\mathbb{K}$	The 'incompressible elasticity tensor', given by $\mathbb{K}[\mathbf{G}_1,\mathbf{G}_2]=\frac{\partial^2 W(\mathbf{D})}{\partial\mathbf{F}^2}[\mathbf{G}_1\mathbf{D},\mathbf{G}_2\mathbf{D}]+p\mathrm{tr}(\mathbf{G}_1\mathbf{G}_2)$, where p is such that $\frac{\partial W(\mathbf{D})}{\partial\mathbf{F}}\mathbf{D}^T\mathbf{n}=p\mathbf{n}$

A.1.4 Chapter 4

$\mathbf{Q}(t)$	The $SO(n)$ matrix rotating an angle t anticlockwise in the (x_1, x_2)-plane
Γ	The half-disk $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1,\ x_1 > 0\}$
Γ_r	The half-disk $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < r,\ x_1 > 0\}$
Γ	The semi-circle $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = r,\ x_1 > 0\}$
Ω	The half-cylinder $\{\mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1,\ x_1 > 0,\ 0 < x_3 < L\}$
$\boldsymbol{\varphi}_{DC_\alpha}$	The two-dimensional double-covering map with fixed volume change α^2, given by $\boldsymbol{\varphi}_{DC_\alpha}(\mathbf{x}) = \frac{\alpha}{\sqrt{2}\lvert\mathbf{x}\rvert}\begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{pmatrix}$
$\widetilde{\boldsymbol{\varphi}}_{DC\gamma}$	The three-dimensional incompressible double-covering map, given by $\widetilde{\boldsymbol{\varphi}}_{DC\gamma}(\mathbf{x}) = \begin{pmatrix} \boldsymbol{\varphi}_{DC_\alpha}(x_1, x_2) \\ \gamma x_3 \end{pmatrix}$

A.1.5 Chapter 5

$\mathbf{D}$	The diagonal 2×2 matrix with λ_i in the (i, i) entry, for $i = 1, 2$ and 0 elsewhere
Γ	The function defined by $\Gamma(\lambda_1, \lambda_2, L) = b\lambda_1^2 L^4 + \left[a\lambda_1^2 + c\lambda_2^2 + 2\alpha\left((\lambda_1\lambda_2)^2 a_2 + \frac{a-1}{\lambda_1\lambda_2}\right)\right] L^2 + d\lambda_2^2$
H_-	The half-space of elements $z \in \mathbb{C}$ with negative imaginary part
$\boldsymbol{\varphi}^c$	The crease forming deformation on H_-, given by $\boldsymbol{\varphi}^c(\mathbf{x}) = h\begin{pmatrix} \mu x\left(1 - \frac{1}{\mu^2x^2 + (1-\mu y)^2}\right) \\ \mu y - \frac{1-\mu y}{\mu^2 x^2 + (1-\mu y)^2} \end{pmatrix}$

A.2 Results for Chapter 2

In this section for results used in Chapter 2, the dimension $n = 3$, and we make use of the notation $\mathbf{n} = \mathbf{e}_3$. Define

$$m_1(\boldsymbol{\tau}, \alpha) = \sqrt{\alpha^2 + \tau_1^2 + \tau_2^2}, \tag{A.1}$$

$$m_3(\boldsymbol{\tau}, \alpha) = \sqrt{\frac{\alpha^2 + (1 + \lambda_2^2\lambda_3^2 s)\tau_1^2 + (1 + \lambda_1^2\lambda_3^2 s)\tau_2^2}{1 + \lambda_1^2\lambda_2^2 s}}, \tag{A.2}$$

where $s = H''(\lambda_1\lambda_2\lambda_3) > 0$.

Lemma A.2.1. *Let* $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$ *be a pure homogeneous deformation given by* (1.39). *Suppose the isotropic stored energy function* Φ *is of the form* (2.25). *Furthermore,* *suppose that* $\alpha \geq 0$ *and* $\boldsymbol{\tau} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *are such that* $m_1(\boldsymbol{\tau}, \alpha) \neq m_3(\boldsymbol{\tau}, \alpha)$*, where* $m_1(\boldsymbol{\tau}, \alpha)$ *and* $m_3(\boldsymbol{\tau}, \alpha)$ *are given by* (A.1) *and* (A.2)*, respectively. Then, the general solution to* (2.24a) *that decays to zero as* $x_3 \to -\infty$ *is given by*

$$\mathbf{z}(x_3) = (a\mathbf{t}_1 + b\mathbf{t}_2)\, e^{m_1(\boldsymbol{\tau},\alpha)x_3} + c\mathbf{t}_3 e^{m_3(\boldsymbol{\tau},\alpha)x_3},$$

where

$$\mathbf{t}_1 = \begin{pmatrix} \frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} \\ 0 \\ -\mathrm{i}\frac{\tau_1}{\lambda_1} \end{pmatrix}, \quad \mathbf{t}_2 = \begin{pmatrix} 0 \\ \frac{m_1(\boldsymbol{\tau},\alpha)}{\lambda_3} \\ -\mathrm{i}\frac{\tau_2}{\lambda_2} \end{pmatrix}, \quad \mathbf{t}_3 = \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \\ \frac{m_3(\boldsymbol{\tau},\alpha)}{\lambda_3} \end{pmatrix}. \tag{A.3}$$

Proof. By Lemma 2.1.13, the roots of $\det(\boldsymbol{\chi}(m)) = 0$ with positive real part are given by the repeated root $m_1(\boldsymbol{\tau}, \alpha)$, and $m_3(\boldsymbol{\tau}, \alpha)$. We have, by assumption, that $m_1(\boldsymbol{\tau}, \alpha) \neq m_3(\boldsymbol{\tau}, \alpha)$. Hence, by standard results for differential equations (see, for example, Coddington and Levinson [CL55, Theorem 4.1 and 6.5]) the general solution to (2.24a) must be of the form

$$\mathbf{z}(x_3) = \mathbf{v}_1 e^{m_1(\boldsymbol{\tau},\alpha)x_3} + \mathbf{v}_2 e^{m_3(\boldsymbol{\tau},\alpha)x_3}, \tag{A.4}$$

where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^3$ are to be determined. Note that

$$\mathrm{Ker}(\boldsymbol{\chi}(m_1(\boldsymbol{\tau}, \alpha))) = \mathrm{Span}\{\mathbf{t}_1, \mathbf{t}_2\},$$

and

$$\mathrm{Ker}(\boldsymbol{\chi}(m_3(\boldsymbol{\tau}, \alpha))) = \mathrm{Span}\{\mathbf{t}_3\},$$

where the vectors $\mathbf{t}_i \in \mathbb{C}^3$, $i = 1, 2, 3$ are given by (A.3). Substituting the general solution (A.4) into (2.24a) gives, without loss of generality, that $\mathbf{v}_1 = a\mathbf{t}_1 + b\mathbf{t}_2$, and $\mathbf{v}_2 = c\mathbf{t}_3$, where $a, b, c \in \mathbb{C}$ are arbitrary constants. □

Lemma A.2.2. *Let* $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$ *be a pure homogeneous deformation given by* (1.39)*. Suppose the isotropic stored energy function* Φ *is of the form* (2.25)*. Furthermore, suppose that* $\alpha \geq 0$ *and* $\boldsymbol{\tau} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *are such that* $m_1(\boldsymbol{\tau}, \alpha) = m_3(\boldsymbol{\tau}, \alpha)$*, where* $m_1(\boldsymbol{\tau}, \alpha)$ *and* $m_3(\boldsymbol{\tau}, \alpha)$ *are given by* (A.1) *and* (A.2)*, respectively. Then, the general solution to* (2.24a) *that decays to zero as* $x_3 \to -\infty$ *is given by*

$$\mathbf{z}(x_3) = \left[\mathbf{A} - \frac{(\lambda_1\lambda_2\lambda_3)^2 s\mathbf{t}_3 \cdot \mathbf{A}}{(2 + \lambda_1^2\lambda_2^2 s) m_1(\boldsymbol{\tau}, \alpha)} x_3 \mathbf{t}_3\right] e^{m_1(\boldsymbol{\tau},\alpha)x_3}, \tag{A.5}$$

where $\mathbf{A} \in \mathbb{C}^3$ *is arbitrary.*

Proof. By Lemma 2.1.13, since $m_1(\boldsymbol{\tau}, \alpha) = m_3(\boldsymbol{\tau}, \alpha)$, the roots of $\det(\boldsymbol{\chi}(m)) = 0$ with positive real part are given by the triply repeated root $m_1(\boldsymbol{\tau}, \alpha)$. In this case, the vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ given by (A.3) are no longer linearly independent, since now

$$\mathrm{i}\frac{\tau_1}{\lambda_1}\mathbf{t}_1 + \mathrm{i}\frac{\tau_2}{\lambda_2}\mathbf{t}_2 = \frac{m_1(\boldsymbol{\tau}, \alpha)}{\lambda_3}\mathbf{t}_3. \tag{A.6}$$

Therefore,

$$\mathrm{Ker}(\boldsymbol{\chi}(m_1(\boldsymbol{\tau}, \alpha))) = \mathrm{Span}\{\mathbf{t}_1, \mathbf{t}_2\} = \mathrm{Ker}(\boldsymbol{\chi}(m_3(\boldsymbol{\tau}, \alpha))).$$

Therefore, by standard results for differential equations (see, for example, Coddington and Levinson [CL55, Theorem 4.1 and 6.5]) the general solution to (2.24a) must be of the form

$$\mathbf{z}(x_3) = (\mathbf{A} + x_3\mathbf{B})\, e^{m_1(\boldsymbol{\tau}, \alpha)x_3}.$$

Substituting this into (2.24a) gives that $\mathbf{B} = c\mathbf{t}_1 + d\mathbf{t}_2$, where $c, d \in \mathbb{C}$, and that $\mathbf{A}, \mathbf{B}$ satisfy

$$\boldsymbol{\chi}(m_1(\boldsymbol{\tau}, \alpha))\mathbf{A} + (2m_1(\boldsymbol{\tau}, \alpha)\mathbf{M} + \mathbf{N} + \mathbf{N}^T)\mathbf{B} = 0.$$

Simplifying this condition results in requiring that c and d satisfy

$$\begin{pmatrix} c \\ d \end{pmatrix} = -\lambda_3 \frac{(\lambda_1\lambda_2\lambda_3)^2 s\mathbf{t}_3 \cdot \mathbf{A}}{(2 + \lambda_1^2\lambda_2^2 s)m_1(\boldsymbol{\tau}, \alpha)^2} \begin{pmatrix} \mathrm{i}\frac{\tau_1}{\lambda_1} \\ \mathrm{i}\frac{\tau_2}{\lambda_2} \end{pmatrix}.$$

Therefore, by (A.6),

$$\mathbf{B} = -\frac{(\lambda_1\lambda_2\lambda_3)^2 s\mathbf{t}_3 \cdot \mathbf{A}}{(2 + \lambda_1^2\lambda_2^2 s)m_1(\boldsymbol{\tau}, \alpha)}\mathbf{t}_3, \tag{A.7}$$

giving, overall, that the general solution to (2.24a) is given by (A.5). □

A.3 Results for Chapter 3

In this section, we work with the extended function W of the incompressible stored energy function W^{inc}. We assume W^{inc}, and therefore W, is isotropic, so there exists a symmetric function Φ satisfying (3.28).

A.3.1 Isotropic materials in two dimensions

In this subsection, the dimension $n = 2$, and λ_1 and λ_2 are such that $\lambda_1\lambda_2 = 1$. For $\alpha > 0$ and $\tau_1 \neq 0$, define

$$a = \lambda_2^2\Psi_{12}, \tag{A.8a}$$

$$b_\alpha = 2(\Phi_{12} + \Theta_{12}) - \lambda_1^2\Phi_{11} - \lambda_2^2\Phi_{22} - \left(\frac{\alpha}{\tau_1}\right)^2, \tag{A.8b}$$

$$c_\alpha = \lambda_1^2\Psi_{12} + \left(\frac{\alpha}{\tau_1}\right)^2, \tag{A.8c}$$

where Φ_{ij}, Ψ_{ij}, and Θ_{ij}, for $i, j = 1, 2$, are given by (2.5).

Lemma A.3.1. *Let Φ satisfy the Legendre-Hadamard condition. Then for any $\alpha > 0$,*

$$b_\alpha < b_0 \leq 2\sqrt{ac_0} < 2\sqrt{ac_\alpha}.$$

Proof. By Lemma 3.3.2,

$$\begin{aligned} 0 &\leq (\lambda_1\sqrt{\Phi_{11}} - \lambda_2\sqrt{\Phi_{22}})^2 = \lambda_1^2\Phi_{11} + \lambda_2^2\Phi_{22} - 2\sqrt{\Phi_{11}\Phi_{22}} \\ &\leq \lambda_1^2\Phi_{11} + \lambda_2^2\Phi_{22} - 2(\Phi_{12} + \Theta_{12}) + 2\Psi_{12} = -b_0 + 2\sqrt{ac_0}. \end{aligned}$$

The claim follows from this and the fact that c_α is strictly increasing and b_α is strictly decreasing in α. □

Lemma A.3.2. *Define $\widetilde{g} : [0, \infty) \to \mathbb{R}$ by*

$$\widetilde{g}(x) = (c_0 + x - a)\sqrt{a(c_0 + x)} + (c_0 + x)(2a - b_0 + x), \tag{A.9}$$

where a, b_α, and c_α are given by (A.8)*. Then, if Φ satisfies the Legendre-Hadamard condition, and if $\widetilde{g}(0) \geq 0$, then $\widetilde{g}$ is strictly increasing on $(0, \infty)$.*

Proof. Note that, by Lemma A.3.1,

$$0 \leq a + (\sqrt{a} - \sqrt{c_0})^2 = 2a + c_0 - 2\sqrt{ac_0} \leq 2a + c_0 - b_0.$$

Suppose

$$\widetilde{g}(0) = (c_0 - a)\sqrt{ac_0} + c_0(2a - b_0) \geq 0.$$

Then for $x > 0$,

$$\begin{aligned}
\widetilde{g}'(x) &= \frac{1}{2\sqrt{a(c_0+x)}}\left[a(3c_0 - a + 3x) + (2c_0 + 4a - 2b_0 + 4x)\sqrt{a(c_0+x)}\right] \\
&> \frac{1}{2\sqrt{a(c_0+x)}}\left[a(3c_0 - a) + (2c_0 + 4a - 2b_0)\sqrt{ac_0}\right] \\
&= \frac{1}{2\sqrt{c_0(c_0+x)}}\left[2c_0\sqrt{ac_0} + (2c_0 + 2a - b_0)c_0 + \widetilde{g}(0)\right] \\
&\geq \frac{1}{2\sqrt{c_0(c_0+x)}}\left[2c_0\sqrt{ac_0} + c_0^2\right] > 0.
\end{aligned}$$

Hence, $\widetilde{g}$ is strictly increasing on $(0, \infty)$. $\square$

For the forthcoming lemmas relating to Agmon's condition for isotropic materials in two dimensions, it is helpful to define, for $\alpha > 0$ and $\tau_1 \in \mathbb{R} \setminus \{0\}$,

$$\widetilde{\boldsymbol{\chi}}(m) = \begin{pmatrix} \lambda_2^2\Psi_{12}m^2 - (\lambda_1^2\Phi_{11} + p_2)\tau_1^2 - \alpha^2 & \mathrm{i}(\lambda_1\lambda_2(\Phi_{12} + \Theta_{12}) + p_2)\tau_1 m & \mathrm{i}\tau_1 \\ \mathrm{i}(\lambda_1\lambda_2(\Phi_{12} + \Theta_{12}) + p_2)\tau_1 m & (\lambda_2^2\Phi_{22} + p_2)m^2 - \lambda_1^2\Psi_{12}\tau_1^2 - \alpha^2 & m \\ \mathrm{i}\tau_1 & m & 0 \end{pmatrix}, \tag{A.10}$$

where Φ_{ij}, Ψ_{ij}, and Θ_{ij}, for $i, j = 1, 2$, are given by (2.5).

Lemma A.3.3. *Let* $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2)$*, and let the tensor* $\mathbb{K}$ *be given by* (3.43)*. Let* $\alpha > 0$*,* $\tau_1 \neq 0$*, and let* $\widetilde{\boldsymbol{\chi}}(m)$ *be given by* (A.10)*. Let* a*,* b_α*, and* c_α *be given by* (A.8)*. Suppose* Φ *is such that the tensor* $\mathbb{K}$ *satisfies the Legendre-Hadamard condition. Then*

1. *if* $a = 0$*, the equation* $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$ *has two roots real roots* $m = \pm m_0$*, where* m_0 *is given by*
$$\frac{m_0}{\tau_1} = \sqrt{-\frac{c_\alpha}{b_\alpha}} \tag{A.11}$$

2. *if* $a > 0$*, the equation* $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$ *has four roots* $m = \pm m_+$ *and* $m = \pm m_-$ *with nonzero real part, where* m_+^2 *and* m_-^2 *are given by*
$$\frac{m_\pm^2}{\tau_1^2} = \frac{-b_\alpha \pm \sqrt{b_\alpha^2 - 4ac_\alpha}}{2a}. \tag{A.12}$$

   In particular, m_+, m_- *satisfy*
$$\frac{m_+^2 m_-^2}{\tau_1^4} = \frac{c}{a}, \tag{A.13}$$
$$\frac{m_+^2}{\tau_1^2} + \frac{m_-^2}{\tau_1^2} = -\frac{b}{a}. \tag{A.14}$$

Proof. Solving the equation $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$ yields a fourth order polynomial in m given by

$$am^4 + b_\alpha \tau_1^2 m^2 + c_\alpha \tau_1^4 = 0. \tag{A.15}$$

1. If $a = 0$, then by Lemma A.3.1, $b_\alpha < 0$. The equation (A.15) is then clearly solved by the two real roots $m = \pm m_0$, where m_0 is given by (A.11).

2. If $a > 0$, (A.15) is a quadratic polynomial in m^2. The quadratic formula gives, overall, four roots $m = \pm m_+$ and $m = \pm m_-$, where m_+ and m_- are given by (3.51). The properties (A.13) and (A.14) follow trivially from (3.51).

   In this case, since $a, c_\alpha > 0$, if either $\mathrm{Re}(m_+) = 0$ or $\mathrm{Re}(m_-) = 0$, then $b_\alpha \geq 0$ and $b_\alpha^2 \geq 4ac_\alpha$. However, this implies that $b_\alpha \geq 2\sqrt{ac_\alpha}$, contradicting Lemma A.3.1. Hence we have that $\mathrm{Re}(m_\pm) \neq 0$.

□

Lemma A.3.4. *Let* $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2)$*, and let the tensor* $\mathbb{K}$ *be given by* (3.43)*. Let* $\alpha > 0$*,* $\tau_1 \neq 0$*, and let* $\widetilde{\boldsymbol{\chi}}(m)$ *be given by* (A.10)*. Define*

$$C_1(m) = \mathrm{i}\tau_1 \left(\left(\Phi_{12} + \Theta_{12} - \lambda_2^2 \Phi_{22} \right) m^2 + \lambda_1^2 \Psi_{12} \tau_1^2 + \alpha^2 \right). \tag{A.16}$$

Suppose the isotropic stored energy function Φ *is such that the tensor* $\mathbb{K}$ *satisfies the Legendre-Hadamard condition. Then*

1. *if* $\Psi_{12} = 0$*,*

$$\mathrm{null}(\widetilde{\boldsymbol{\chi}}(m_0)) = \mathrm{Span}\left\{ \begin{pmatrix} -m_0 \\ \mathrm{i}\tau_1 \\ \frac{C_1(m_0)}{m_0} \end{pmatrix} \right\}, \tag{A.17}$$

   where m_0 *is given by* (A.11)*;*

2. *if* $\Psi_{12} > 0$*,*

$$\mathrm{null}(\widetilde{\boldsymbol{\chi}}(m_\pm)) = \mathrm{Span}\left\{ \begin{pmatrix} -m_\pm \\ \mathrm{i}\tau_1 \\ \frac{C_1(m_\pm)}{m_\pm} \end{pmatrix} \right\}, \tag{A.18}$$

   where $m_\pm$ *is given by* (3.51)*.*

Proof. Note that

$$\mathrm{Cof}\widetilde{\boldsymbol{\chi}}(m) = \begin{pmatrix} -m^2 & \mathrm{i}\tau_1 m & C_1(m) \\ \mathrm{i}\tau_1 m & \tau_1^2 & C_2(m) \\ C_1(m) & C_2(m) & C_3(m) \end{pmatrix}$$

where $C_1(m)$ is given by (A.16), and

$$C_2(m) = -m\left(\lambda_2^2\Psi_{12}m^2 + \left(\Phi_{12}+\Theta_{12}-\lambda_1^2\Phi_{11}\right)\tau_1^2-\alpha^2\right), \tag{A.19}$$

$$\begin{aligned} C_3(m) = {} & \left(\lambda_2^2\Psi_{12}m^2 - \lambda_1^2\Phi_{11}\tau_1^2-\alpha^2\right)\left(\lambda_2^2\Phi_{22}m^2-\lambda_1^2\Psi_{12}\tau_1^2-\alpha^2\right) \\ & + (\Phi_{12}+\Theta_{12})^2\tau_1^2m^2. \end{aligned} \tag{A.20}$$

Suppose that m is a root of (A.15). Then we have the properties

$$C_2(m) = -\frac{\mathrm{i}\tau_1 C_1(m)}{m}, \quad C_3(m) = \frac{C_1(m)C_2(m)}{\mathrm{i}\tau_1 m}.$$

So

$$\mathrm{Cof}\widetilde{\boldsymbol{\chi}}(m) = \begin{pmatrix} \mathrm{i}m \\ \tau_1 \\ -\frac{\mathrm{i}C_1(m)}{m} \end{pmatrix} \otimes \begin{pmatrix} \mathrm{i}m \\ \tau_1 \\ -\frac{\mathrm{i}C_1(m)}{m} \end{pmatrix}, \tag{A.21}$$

and in particular, $\mathrm{rank}(\mathrm{Cof}\widetilde{\boldsymbol{\chi}}(m)) = 1$. We know that $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(m)) \leq 2$. If $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(m)) \leq 1$, then each 2×2 minor of $\widetilde{\boldsymbol{\chi}}(m)$ is zero, since each of its rows are linearly dependant. This means $\mathrm{Cof}\widetilde{\boldsymbol{\chi}}(m) \equiv 0$, which is not possible. So we conclude that $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(m)) = 2$, and so, by the rank-nullity theorem, $\mathrm{nullity}(\widetilde{\boldsymbol{\chi}}(m)) = 1$. The null space of $\widetilde{\boldsymbol{\chi}}(m)$ is spanned by any one of the columns of $\mathrm{Cof}\widetilde{\boldsymbol{\chi}}(m)$. Hence, by (A.21), we have that

$$\mathrm{null}(\widetilde{\boldsymbol{\chi}}(m)) = \mathrm{Span}\left\{\begin{pmatrix} -m \\ \mathrm{i}\tau_1 \\ \frac{C_1(m)}{m} \end{pmatrix}\right\}.$$

1. If $\Psi_{12} = 0$, then $a = 0$. Since $m = m_0$ is a root of (A.15), we have (A.17).

2. If $\Psi_{12} > 0$, then $a > 0$. Since $m = m_\pm$ is a root of (A.15), we have (A.18).

□

Lemma A.3.5. *Let* $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2)$*, and let the tensor* $\mathbb{K}$ *be given by* (3.43)*. Let* $\alpha > 0$*,* $\tau_1 \neq 0$*, and let* $\widetilde{\boldsymbol{\chi}}(m)$ *be given by* (A.10)*. Suppose the isotropic stored energy function* Φ *is such that the tensor* $\mathbb{K}$ *satisfies the Legendre-Hadamard condition, and suppose* $\Psi_{12} = 0$*. Then the general solution to* (1.55a) *and* (1.55b) *that decays to zero as* $t \to -\infty$ *is given by*

$$\mathbf{z}(t) = k\begin{pmatrix} -m_0 \\ \mathrm{i}\tau_1 \end{pmatrix} e^{m_0 t},$$

$$q(t) = k\frac{C_1(m_0)}{m_0}e^{m_0 t},$$

where m_0 is given by (A.11)*,* C_1 *is given by* (A.16)*, and* $k \in \mathbb{C}$ *is an arbitrary constant.*

Proof. To solve (1.55a) and (1.55b), we seek nonzero solutions of the form $(\mathbf{z}(t), q(t)) = (\mathbf{A}, \mu)e^{mt}$, where $\mathbf{A} \in \mathbb{C}^2$ and $\mu \in \mathbb{C}$. This requires that $(\mathbf{A}, \mu) \neq (\mathbf{0}, 0)$ satisfy

$$\widetilde{\boldsymbol{\chi}}(m) \begin{pmatrix} A_1 \\ A_2 \\ \mu \end{pmatrix} = 0,$$

where $\widetilde{\boldsymbol{\chi}}(m)$ is given by (A.10). Note that $\Psi_{12} = 0$ implies $a = 0$. By Lemma A.3.3, there exist two real roots $m = \pm m_0$ to the equation $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$, given by (3.58). Hence, the system (1.55a), (1.55b) is order two, so by standard results for differential equations (see, for example, Coddington and Levinson [CL55, Theorem 4.1 and 6.5]) the general solution to (1.55a), (1.55b) that decays to zero as $t \to -\infty$ is given by[1]

$$(\mathbf{z},\ q) = (\mathbf{A}e^{m_0 t},\ \mu e^{m_0 t}),$$

where $\mathbf{A}$ and μ satisfy

$$\widetilde{\boldsymbol{\chi}}(m_0) \begin{pmatrix} A_1 \\ A_2 \\ \mu \end{pmatrix} = 0. \tag{A.22}$$

By Lemma A.3.4, we have that $\mathbf{A}$ and μ are given by

$$\begin{pmatrix} A_1 \\ A_2 \\ \mu \end{pmatrix} = k \begin{pmatrix} -m_0 \\ \mathrm{i}\tau_1 \\ \frac{C_1(m_0)}{m_0} \end{pmatrix},$$

where $k \in \mathbb{C}$ is an arbitrary constant. □

Lemma A.3.6. *Let* $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2)$*, and let the tensor* $\mathbb{K}$ *be given by* (3.43)*. Let* $\alpha > 0$*,* $\tau_1 \neq 0$*, and let* $\widetilde{\boldsymbol{\chi}}(m)$ *be given by* (A.10)*. Suppose the isotropic stored energy function* Φ *is such that the tensor* $\mathbb{K}$ *satisfies the Legendre-Hadamard condition, and that* $\Psi_{12} > 0$*. Suppose further that* α *and* τ_1 *are such that* $b_\alpha^2 \neq 4ac_\alpha$*, where* a*,* b_α*, and* c_α *are given by* (A.8)*. Then the general solution to* (1.55a) *and* (1.55b) *that decays to zero as* $t \to -\infty$ *is given by*

$$\begin{aligned} \mathbf{z}(t) &= k \begin{pmatrix} -m_+^2 \\ \mathrm{i}\tau_1 m_+ \end{pmatrix} e^{m_+ t} + l \begin{pmatrix} -m_-^2 \\ \mathrm{i}\tau_1 m_- \end{pmatrix} e^{m_- t}, \\ q(t) &= kC_1(m_+)e^{m_+ t} + lC_1(m_-)e^{m_- t}, \end{aligned} \tag{A.23}$$

[1] We omit the other solution corresponding to the root $-m_0$ for decaying solutions.

where $m_\pm$ are given by (A.12)*, C_1 is given by* (A.16)*, and $k, l \in \mathbb{C}$ are arbitrary constants.*

Proof. To solve (1.55a) and (1.55b), we seek nonzero solutions of the form $(\mathbf{z}(t), q(t)) = (\mathbf{A}, \mu)e^{mt}$, where $\mathbf{A} \in \mathbb{C}^2$ and $\mu \in \mathbb{C}$. This requires that $(\mathbf{A}, \mu) \neq \mathbf{0}$ satisfy

$$\widetilde{\boldsymbol{\chi}}(m) \begin{pmatrix} A_1 \\ A_2 \\ \mu \end{pmatrix} = 0,$$

where $\widetilde{\boldsymbol{\chi}}(m)$ is given by (A.10). Note that $\Psi_{12} > 0$ implies $a > 0$. By Lemma A.3.3, there exist four roots to the equation $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$, $m = \pm m_+$ and $m = \pm m_-$ given by (3.51), each with nonzero real part. Since the roots $m_\pm$ come in plus or minus pairs, and $\mathrm{Re}(m_\pm) \neq 0$, without loss of generality we have that $\mathrm{Re}(m_\pm) > 0$. Furthermore, by the assumption that $b_\alpha^2 \neq 4ac_\alpha$, we have $m_+ \neq m_-$. Then the system (1.55a), (1.55b) is order four, so by standard results for differential equations (see, for example, Coddington and Levinson [CL55, Theorem 4.1 and 6.5]) the general solution to (1.55a), (1.55b) that decays to zero as $t \to -\infty$ is given by

$$(\mathbf{z},\ q) = (\mathbf{A}e^{m_+t} + \mathbf{B}e^{m_-t},\ \mu e^{m_+t} + \nu e^{m_-t}),$$

where $\mathbf{A}, \mathbf{B}, \mu, \nu$ satisfy

$$\widetilde{\boldsymbol{\chi}}(m_+) \begin{pmatrix} A_1 \\ A_2 \\ \mu \end{pmatrix} = 0, \qquad \widetilde{\boldsymbol{\chi}}(m_-) \begin{pmatrix} B_1 \\ B_2 \\ \nu \end{pmatrix} = 0. \tag{A.24}$$

By Lemma A.3.4, since $a > 0$, we have that $\mathbf{A}$, $\mathbf{B}$, μ, and ν are given by

$$\begin{pmatrix} A_1 \\ A_2 \\ \mu \end{pmatrix} = k \begin{pmatrix} -m_+^2 \\ \mathrm{i}\tau_1 m_+ \\ C_1(m_+) \end{pmatrix}, \qquad \begin{pmatrix} B_1 \\ B_2 \\ \nu \end{pmatrix} = l \begin{pmatrix} -m_-^2 \\ \mathrm{i}\tau_1 m_- \\ C_1(m_-) \end{pmatrix},$$

where $k, l \in \mathbb{C}$ are arbitrary. This form for $\mathbf{A}$, $\mathbf{B}$, μ, and u gives that the general solution to (1.55a) and (1.55b) in the case $b_\alpha^2 \neq 4ac_\alpha$ is given by (A.23). $\square$

Lemma A.3.7. *Let $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2)$, and let the tensor $\mathbb{K}$ be given by* (3.43)*. Let $\alpha > 0$, $\tau_1 \neq 0$, and let $\widetilde{\boldsymbol{\chi}}(m)$ be given by* (A.10)*. Suppose the isotropic stored energy function Φ is such that the tensor $\mathbb{K}$ satisfies the Legendre-Hadamard condition, and that $\Psi_{12} > 0$. Suppose further that α and τ_1 are such that $b_\alpha^2 = 4ac_\alpha$, where a, b_α, and c_α are given by* (A.8)*. Then the general solution to* (1.55a) *and* (1.55b) *that decays to* *zero as* $t \to -\infty$ *is given by*

$$\begin{aligned}\mathbf{z}(t) &= \left[k_1\begin{pmatrix}-1\\0\end{pmatrix}+k_2\begin{pmatrix}-m_+\\ \mathrm{i}\tau_1\end{pmatrix}+k_1t\begin{pmatrix}-m_+\\ i\tau_1\end{pmatrix}\right]e^{m_+t}\\ q(t) &= \left[k_1\left(\mathrm{i}b_\alpha\tau_1+\frac{1}{m_+^2}C_1(m_+)\right)+\frac{k_2}{m_+}C_1(m_+)+\frac{k_1t}{m_+}C_1(m_+)\right]e^{m_+t}\end{aligned} \tag{A.25}$$

where $C_1(m)$ *is given by* (A.16), *and* $k, l \in \mathbb{C}$ *are arbitrary constants.*

Proof. Similar to Lemma A.3.6, we solve (1.55a) and (1.55b) by seeking nonzero solutions of the form $(\mathbf{z}(t), q(t)) = (\mathbf{A}, \mu)e^{mx_3}$, where $\mathbf{A} \in \mathbb{C}^2$ and $\mu \in \mathbb{C}$. This requires that $(\mathbf{A}, \mu) \neq \mathbf{0}$ satisfy

$$\widetilde{\boldsymbol{\chi}}(m)\begin{pmatrix}A_1\\A_2\\ \mu\end{pmatrix}=0,$$

where $\widetilde{\boldsymbol{\chi}}(m)$ is given by (A.10). Note that $\Psi_{12} > 0$ implies $a > 0$. By Lemma A.3.3, there exist four roots to the equation $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$, $m = \pm m_+$ and $m = \pm m_-$ given by (3.51). Since $b_\alpha^2 = 4ac_\alpha$, we have repeated roots $m = m_+ = m_-$ and $m = -m_+ = -m_-$. Since either root $m_\pm$ comes in a plus or minus pair, and $\mathrm{Re}(m_\pm) \neq 0$, without loss of generality, $\mathrm{Re}(m_\pm) > 0$. Hence, the system (1.55a), (1.55b) is order four, so by standard results for differential equations (see for example, Coddington and Levinson [CL55, Theorem 4.1 and 6.5]), the general solution takes the form

$$(\mathbf{z}(t), q(t)) = ((\mathbf{A}+t\mathbf{B})e^{m_+t}, (\mu+t\nu)e^{m_+t}).$$

By substituting this into (1.55a) and (1.55b), we have, for some arbitrary $k_1, k_2 \in \mathbb{C}$, that $\mathbf{A}$, $\mathbf{B}$, μ, and ν are given by

$$\begin{aligned}\begin{pmatrix}\mathbf{A}\\ \mu\end{pmatrix} &= k_1\begin{pmatrix}-1\\0\\ \mathrm{i}b_\alpha\tau_1+\frac{1}{m_+^2}C_1(m_+)\end{pmatrix}+k_2\begin{pmatrix}-m_+\\ \mathrm{i}\tau_1\\ \frac{1}{m_+}C_1(m_+)\end{pmatrix},\\ \begin{pmatrix}\mathbf{B}\\ \nu\end{pmatrix} &= k_1\begin{pmatrix}-m_+\\ \mathrm{i}\tau_1\\ \frac{1}{m_+}C_1(m_+)\end{pmatrix},\end{aligned}$$

where $C_1(m)$ is given by (A.16) from the nondegenerate case. □

A.3.2 Neo-Hookean materials in three dimensions

In $n = 3$ dimensions, let Φ be of the 'neo-Hookean' form (3.70). Define, for $\alpha > 0$ and $\boldsymbol{\tau} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ orthogonal to $\mathbf{n} = \mathbf{e}_3$,

$$\widetilde{\boldsymbol{\chi}}(m) = \left(\begin{array}{ccc|c} & & & \mathrm{i}\tau_1 \\ & (\lambda_3^2 m^2 - \lambda_1^2\tau_1^2 - \lambda_2^2\tau_2^2 - \alpha^2)\mathbb{1} & & \mathrm{i}\tau_2 \\ & & & m \\ \hline \mathrm{i}\tau_1 & \mathrm{i}\tau_2 & m & 0 \end{array}\right) + p_3 \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ m \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ m \\ 0 \end{pmatrix}. \tag{A.26}$$

Lemma A.3.8. *Let* $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$*, let* $\alpha > 0$*,* $\boldsymbol{\tau} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ *be orthogonal to* $\mathbf{n}$*, and let* $\widetilde{\boldsymbol{\chi}}(m)$ *be given by* (A.26)*. Then equation* $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$ *has six roots, two given by* $\pm|\boldsymbol{\tau}|$*, and the remaining four given by the repeated roots* $\pm\sigma(\boldsymbol{\tau}, \alpha)$*, where* $\sigma(\boldsymbol{\tau}, \alpha)$ *is defined by*

$$\sigma(\boldsymbol{\tau}, \alpha) := \frac{1}{\lambda_3}\sqrt{\lambda_1^2\tau_1^2 + \lambda_2^2\tau_2^2 + \alpha^2}. \tag{A.27}$$

Proof. A simple calculation gives that $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$ if and only if

$$(m^2 - \tau_1^2 - \tau_2^2)(\lambda_3^2 m^2 - \lambda_1^2\tau_1^2 - \lambda_2^2\tau_2^2 - \alpha^2)^2 = 0,$$

proving the lemma. $\square$

Lemma A.3.9. *Let* $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$*, let* $\alpha > 0$*,* $\boldsymbol{\tau} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ *be orthogonal to* $\mathbf{n}$*, and let* $\widetilde{\boldsymbol{\chi}}(m)$ *be given by* (A.26)*. Denote* $\sigma(\boldsymbol{\tau}, \alpha)$ *by* (A.27)*. Then*

if $\sigma(\boldsymbol{\tau}, \alpha) \neq |\boldsymbol{\tau}|$,

$$\operatorname{null}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)) = \operatorname{Span}\left\{\begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ |\boldsymbol{\tau}| \\ \lambda_3^2(\sigma(\boldsymbol{\tau}, \alpha)^2 - |\boldsymbol{\tau}|^2) \end{pmatrix}\right\}, \tag{A.28}$$

and

$$\operatorname{null}(\widetilde{\boldsymbol{\chi}}(\sigma(\boldsymbol{\tau}, \alpha))) = \operatorname{Span}\left\{\begin{pmatrix} \mathrm{i}\tau_2\sigma(\boldsymbol{\tau}, \alpha) \\ \mathrm{i}\tau_1\sigma(\boldsymbol{\tau}, \alpha) \\ 2\tau_1\tau_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\mathrm{i}\sigma(\boldsymbol{\tau}, \alpha)\tau_1 \\ \mathrm{i}\sigma(\boldsymbol{\tau}, \alpha)\tau_2 \\ \tau_2^2 - \tau_1^2 \\ 0 \end{pmatrix}\right\}, \tag{A.29}$$

if $\sigma(\boldsymbol{\tau},\alpha)=|\boldsymbol{\tau}|$,

$$\mathrm{null}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)) = \mathrm{Span}\left\{\begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ |\boldsymbol{\tau}| \\ 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i}\tau_2|\boldsymbol{\tau}| \\ \mathrm{i}\tau_1|\boldsymbol{\tau}| \\ 2\tau_1\tau_2 \\ 0 \end{pmatrix}\right\}, \tag{A.30}$$

Proof. By (A.26), we can write

$$\widetilde{\boldsymbol{\chi}}(\sigma(\boldsymbol{\tau},\alpha)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{p_3} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{p_3} \end{pmatrix} + p_3 \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ m \\ \frac{1}{p_3} \end{pmatrix} \otimes \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ m \\ \frac{1}{p_3} \end{pmatrix}, \tag{A.31}$$

so $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(\sigma(\boldsymbol{\tau},\alpha)))=2$. Furthermore, after a lengthy calculation, we have that

$$\mathrm{Cof}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)) = R_{\boldsymbol{\tau},\alpha}(|\boldsymbol{\tau}|) \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ |\boldsymbol{\tau}| \\ -R_{\boldsymbol{\tau},\alpha}(|\boldsymbol{\tau}|) \end{pmatrix} \otimes \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ |\boldsymbol{\tau}| \\ -R_{\boldsymbol{\tau},\alpha}(|\boldsymbol{\tau}|) \end{pmatrix}, \tag{A.32}$$

where $R_{\boldsymbol{\tau},\alpha}(m)=\lambda_3^2(m^2-\sigma(\boldsymbol{\tau},\alpha)^2)$.

If $\sigma(\boldsymbol{\tau},\alpha)\neq|\boldsymbol{\tau}|$: We know that $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|))\leq 3$, and by (A.32), $\mathrm{rank}(\mathrm{Cof}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)))=1$. If $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|))\leq 2$, then each 3×3 minor of $\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)$ is zero, since at least two of each of its rows are linearly dependant. However, this means $\mathrm{Cof}\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)\equiv 0$, which is not true. So we conclude that $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|))=3$, and so, by the rank-nullity theorem, $\mathrm{nullity}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|))=1$. By the property that $\mathbf{F}\mathrm{Cof}(\mathbf{F})^T=\mathbf{0}$ for any singular matrix $\mathbf{F}$, and (A.32), we have (A.28).

We have that $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(\sigma(\boldsymbol{\tau},\alpha)))=2$, so by rank-nullity, $\mathrm{nullity}(\widetilde{\boldsymbol{\chi}}(\sigma(\boldsymbol{\tau},\alpha)))=2$. One can readily check that the vectors in (A.29) are in the null space of $\widetilde{\boldsymbol{\chi}}(\sigma(\boldsymbol{\tau},\alpha))$, and are linearly independent by the assumption that $\sigma(\boldsymbol{\tau},\alpha)\neq|\boldsymbol{\tau}|$. Hence, we have (A.29).

If $\sigma(\boldsymbol{\tau},\alpha)=|\boldsymbol{\tau}|$: By (A.31) and the fact that $\sigma(\boldsymbol{\tau},\alpha)=|\boldsymbol{\tau}|$, we have that $\mathrm{rank}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|))=2$, so by rank-nullity, $\mathrm{nullity}(\mathrm{Cof}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)))=2$. One can readily check that the vectors in (A.30) are in the null space of $\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)$, and are linearly independent. Hence we have (A.30).

$\square$

Lemma A.3.10. *Let* $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$*, let* $\alpha > 0$*,* $\boldsymbol{\tau} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ *be orthogonal to* $\mathbf{n}$*, and let* $\widetilde{\mathbf{M}}$*,* $\widetilde{\mathbf{N}}$ *and* $\widetilde{\mathbf{P}}$ *be given by* (3.74)*,* (3.75)*, and* (3.76)*, respectively. Suppose that* α *and* $\boldsymbol{\tau}$ *are such that* $\sigma(\boldsymbol{\tau}, \alpha) \neq |\boldsymbol{\tau}|$*, where* $\sigma(\boldsymbol{\tau}, \alpha)$ *is given by* (A.27)*. Then the general solution to* (1.55a) *and* (1.55b) *that decays to zero as* $t \to -\infty$ *is given by*

$$(\mathbf{z}, q) = (\mathbf{A}e^{|\boldsymbol{\tau}|t} + (\mathbf{B} + \mathbf{C})e^{\sigma(\boldsymbol{\tau},\alpha)t}, \mu e^{|\boldsymbol{\tau}|t} + (\nu + \rho)e^{\sigma(\boldsymbol{\tau},\alpha)t}),$$

where

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \mu \end{pmatrix} = k_1 \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ |\boldsymbol{\tau}| \\ \lambda_3^2(\sigma(\boldsymbol{\tau}, \alpha)^2 - |\boldsymbol{\tau}|^2) \end{pmatrix}, \tag{A.33}$$

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \nu \end{pmatrix} = k_2 \begin{pmatrix} \mathrm{i}\tau_2\sigma(\boldsymbol{\tau}, \alpha) \\ \mathrm{i}\tau_1\sigma(\boldsymbol{\tau}, \alpha) \\ 2\tau_1\tau_2 \\ 0 \end{pmatrix}, \tag{A.34}$$

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \rho \end{pmatrix} = k_3 \begin{pmatrix} -\mathrm{i}\sigma(\boldsymbol{\tau}, \alpha)\tau_1 \\ \mathrm{i}\sigma(\boldsymbol{\tau}, \alpha)\tau_2 \\ \tau_2^2 - \tau_1^2 \\ 0 \end{pmatrix}, \tag{A.35}$$

and $k_1, k_2, k_3 \in \mathbb{C}$ *are arbitrary.*

Proof. We seek solutions to (1.55) of the form $(\mathbf{z}(t), q(t)) = (\mathbf{v}e^{mt}, re^{mt})$, where $\mathbf{v} \in \mathbb{C}$ and $r \in \mathbb{C}$. Substituting this into (1.55a) and (1.55b) leads to the requirement that m, $\mathbf{v}$, and r satisfy

$$\widetilde{\boldsymbol{\chi}}(m) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ r \end{pmatrix} = 0, \tag{A.36}$$

where $\widetilde{\boldsymbol{\chi}}(m)$ is given by (A.26). There exist nonzero $\mathbf{v}$, r that satisfy (A.36) if and only if $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$. By Lemma A.3.8, there exist six roots to this characteristic equation, two given by $\pm|\boldsymbol{\tau}|$, and the remaining four given by the repeated roots $\pm\sigma(\boldsymbol{\tau}, \alpha)$. Since, by the assumption that $\sigma(\boldsymbol{\tau}, \alpha) \neq |\boldsymbol{\tau}|$, by standard results for differential equations (see for example, Coddington and Levinson [CL55, Theorem 4.1 and 6.5]), the general solution to (1.55a) and (1.55b) that decays to zero as $t \to -\infty$ is of the form

$$(\mathbf{z}, q) = (\mathbf{v}_1 e^{|\boldsymbol{\tau}|t} + \mathbf{v}_2 e^{\sigma(\boldsymbol{\tau},\alpha)t}, r_1 e^{|\boldsymbol{\tau}|t} + r_2 e^{\sigma(\boldsymbol{\tau},\alpha)t}). \tag{A.37}$$

Since (A.37) satisfies (1.55a) and (1.55b), we find that

$$\begin{pmatrix} \mathbf{v}_1 \\ r_1 \end{pmatrix} \in \operatorname{null}(\widetilde{\boldsymbol{\chi}}(|\tau|)), \qquad \begin{pmatrix} \mathbf{v}_2 \\ r_2 \end{pmatrix} \in \operatorname{null}(\widetilde{\boldsymbol{\chi}}(\sigma(\boldsymbol{\tau}, \alpha))).$$

By Lemma A.3.9, since $\sigma(\boldsymbol{\tau}, \alpha) \neq |\boldsymbol{\tau}|$, without loss of generality, $\mathbf{v}_1 = \mathbf{A}$, $r_1 = \mu$, $\mathbf{v}_2 = \mathbf{B} + \mathbf{C}$, and $r_2 = \nu + \rho$, where $(\mathbf{A}, \mu)$, $(\mathbf{B}, \nu)$, and $(\mathbf{C}, \rho)$ are given by (A.33), (A.34), and (A.35), respectively. $\square$

Lemma A.3.11. *Let* $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$*, let* $\alpha > 0$*,* $\boldsymbol{\tau} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ *be orthogonal to* $\mathbf{n}$*, and let* $\widetilde{\mathbf{M}}$*,* $\widetilde{\mathbf{N}}$ *and* $\widetilde{\mathbf{P}}$ *be given by* (3.74)*,* (3.75)*, and* (3.76)*, respectively. Suppose that* α *and* $\boldsymbol{\tau}$ *are such that* $\sigma(\boldsymbol{\tau}) = |\boldsymbol{\tau}|$*, where* $\sigma(\boldsymbol{\tau})$ *is given by* (A.27)*. Then the general solution to* (1.55a) *and* (1.55b) *that decays to zero as* $t \to -\infty$ *is given by*

$$(\mathbf{z}, q) = ((\mathbf{U} + t\mathbf{V})e^{|\boldsymbol{\tau}|t}, (u + tv)e^{|\boldsymbol{\tau}|t}),$$

where

$$\begin{pmatrix} \mathbf{U} \\ u \end{pmatrix} = k_1 \begin{pmatrix} |\boldsymbol{\tau}| \\ 0 \\ -\mathrm{i}\tau_1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2\lambda_3^2|\boldsymbol{\tau}| \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ |\boldsymbol{\tau}| \\ -\mathrm{i}\tau_2 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{V} \\ v \end{pmatrix} = k_2 \begin{pmatrix} \mathrm{i}\tau_1 \\ \mathrm{i}\tau_2 \\ |\boldsymbol{\tau}| \\ 0 \end{pmatrix},$$

and $k_1, k_2, k_3 \in \mathbb{C}$ *are arbitrary.*

Proof. Similar to Lemma A.3.10, we seek solutions to (1.55) of the form $(\mathbf{z}(t), q(t)) = (\mathbf{v}e^{mt}, re^{mt})$, where $\mathbf{v} \in \mathbb{C}$ and $r \in \mathbb{C}$. Substituting this into (1.55a) and (1.55b) leads to the requirement that m, $\mathbf{v}$, and r satisfy (A.36). There exist nonzero $\mathbf{v}$, r that satisfy (A.36) if and only if $\det(\widetilde{\boldsymbol{\chi}}(m)) = 0$. By Lemma A.3.8, there exist six roots to this characteristic equation, this time given by the triply repeated root $\pm|\boldsymbol{\tau}|$ due to the assumption that $\sigma(\boldsymbol{\tau}) = |\boldsymbol{\tau}|$. By standard results for differential equations (see for example, Coddington and Levinson [CL55, Theorem 4.1 and 6.5]), the general solution to (1.55a) and (1.55b) that decays to zero as $t \to -\infty$ is of the form

$$(\mathbf{z}, q) = ((\mathbf{v}_1 + t\mathbf{v}_2)e^{|\boldsymbol{\tau}|t}, (r_1 + r_2 t)e^{|\boldsymbol{\tau}|t}). \tag{A.38}$$

By substituting (A.38) into (1.55a) and (1.55b), we require that

$$\begin{pmatrix} \mathbf{v}_2 \\ r_2 \end{pmatrix} \in \mathrm{null}(\widetilde{\boldsymbol{\chi}}(|\boldsymbol{\tau}|)),$$

so that, without loss of generality, $\mathbf{v}_2 = \mathbf{V}$ and $r_2 = v = 0$. In addition, we require that

$$\begin{pmatrix} -p\tau_1^2 & -p\tau_1\tau_2 & \mathrm{i}p\tau_1|\boldsymbol{\tau}| & \mathrm{i}\tau_1 \\ -p\tau_1\tau_2 & -p\tau_2^2 & \mathrm{i}p\tau_2|\boldsymbol{\tau}| & \mathrm{i}\tau_2 \\ \mathrm{i}p\tau_1|\boldsymbol{\tau}| & \mathrm{i}p\tau_2|\boldsymbol{\tau}| & p|\boldsymbol{\tau}|^2 & |\boldsymbol{\tau}| \\ \mathrm{i}\tau_1 & \mathrm{i}\tau_2 & |\boldsymbol{\tau}| & 0 \end{pmatrix} \mathbf{v}_1 + \left(\begin{array}{ccc|c} & & & 0 \\ & 2|\boldsymbol{\tau}|\mathbf{M} + \mathbf{N} + \mathbf{N}^T & & 0 \\ & & & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right) \mathbf{v}_2 = \mathbf{0}. \tag{A.39}$$

A tedious calculation in simplifying (A.39) concludes that $\mathbf{v}_1 = \mathbf{U}$ and $r_1 = u$. $\square$

A.4 Results for Chapter 4

Lemma A.4.1. *A deformation $\boldsymbol{\varphi}$ satisfies the equilibrium equations* (1.17) *if and only if $\boldsymbol{\varphi}$ satisfies the system of partial differential equations given by*

$$\frac{\partial}{\partial R}T_{11}^{(R)} + \frac{1}{R}\frac{\partial}{\partial \Theta}T_{12}^{(R)} + \frac{\partial}{\partial Z}T_{13}^{(R)} + \tfrac{1}{R}\left(T_{11}^{(R)} - T_{22}^{(R)}\right) = 0 \tag{A.40a}$$

$$\frac{\partial}{\partial R}T_{12}^{(R)} + \frac{1}{R}\frac{\partial}{\partial \Theta}T_{22}^{(R)} + \frac{\partial}{\partial Z}T_{23}^{(R)} + \tfrac{2}{R}\left(T_{12}^{(R)}\right) = 0 \tag{A.40b}$$

$$\frac{\partial}{\partial R}T_{13}^{(R)} + \frac{1}{R}\frac{\partial}{\partial \Theta}T_{23}^{(R)} + \frac{\partial}{\partial Z}T_{33}^{(R)} + \tfrac{1}{R}\left(T_{13}^{(R)}\right) = 0, \tag{A.40c}$$

Proof. The equilibrium equations in the deformed configuration takes the form

$$\frac{\partial}{\partial \varphi_j}T_{ij} = 0, \qquad \text{for } i = 1, 2, 3,$$

Let (R, Θ, Z) denote radial coordinates of the deformed configuration, so that $\varphi_1 = R\cos(\Theta)$, $\varphi_2 = R\sin(\Theta)$, and $\varphi_3 = Z$. By collectively denoting the deformed cylindrical coordinates by $\boldsymbol{\psi} = (R, \Theta, Z)$, we have

$$\begin{aligned}\frac{\partial}{\partial \varphi_j} T_{ij} &= \frac{\partial \psi_k}{\partial \varphi_j} \frac{\partial}{\partial \psi_k} T_{ij} \\ &= \frac{\partial R}{\partial \varphi_j} \frac{\partial}{\partial R}(T_{ij}) + \frac{\partial \Theta}{\partial \varphi_j} \frac{\partial}{\partial \Theta}(T_{ij}) + \frac{\partial Z}{\partial \varphi_j} \frac{\partial}{\partial Z}(T_{ij}).\end{aligned} \tag{A.41}$$

To find the partial derivatives $\frac{\partial \psi_k}{\partial \varphi_j}$, it is helpful to note that each derivative is the (k, j)th entry of the jacobian inverse

$$\frac{\partial(\varphi_1, \varphi_2, \varphi_3)}{\partial(R, \Theta, Z)} = \begin{pmatrix} \cos(\Theta) & -R\sin(\Theta) & 0 \\ \sin(\Theta) & R\cos(\Theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{Q}(\Theta)\,\mathrm{diag}(1, R, 1).$$

Therefore,

$$\left(\frac{\partial \psi_k}{\partial \varphi_j}\right) = \left(\frac{\partial(\varphi_1, \varphi_2, \varphi_3)}{\partial(R, \Theta, Z)}\right)^{-1} = \mathrm{diag}\left(1, \tfrac{1}{R}, 1\right) \mathbf{Q}(\Theta)^T \tag{A.42}$$

We now write $\mathbf{T} = \mathbf{Q}(\Theta)\mathbf{T}^{(R)}\mathbf{Q}(\Theta)^T$, where $\mathbf{T}^{(R)}$ is the *radial* Cauchy Stress Tensor. Then we have

$$\begin{aligned}\frac{\partial}{\partial \varphi_j} T_{ij} &= \frac{\partial \psi_k}{\partial \varphi_j} \frac{\partial}{\partial \psi_k} T_{ij} \\ &= \left(\mathrm{diag}\left(1, \tfrac{1}{R}, 1\right) \mathbf{Q}(\Theta)^T\right)_{kj} \frac{\partial}{\partial \psi_k} T_{ij} \\ &= \left(\mathrm{diag}(1, \tfrac{1}{R}, 1)\mathbf{Q}(\Theta)^T\right)_{1j} Q(\Theta)_{iq} \frac{\partial}{\partial R} T^{(R)}_{qs} Q(\Theta)_{js} \\ &\quad + \left(\mathrm{diag}(1, \tfrac{1}{R}, 1)\mathbf{Q}(\Theta)^T\right)_{2j} \Big[\frac{\partial}{\partial \Theta}(Q(\Theta)_{iq}) T^{(R)}_{qs} Q(\Theta)_{js} \\ &\quad + Q(\Theta)_{iq} \frac{\partial}{\partial \Theta}(T^{(R)}_{qs}) Q(\Theta)_{js} + Q(\Theta)_{iq} T^{(R)}_{qs} \frac{\partial}{\partial \Theta}(Q(\Theta)_{js})\Big] \\ &\quad + \left(\mathrm{diag}(1, \tfrac{1}{R}, 1)\mathbf{Q}(\Theta)^T\right)_{3j} Q(\Theta)_{iq} \frac{\partial}{\partial Z} T^{(R)}_{qs} Q(\Theta)_{js} \\ &= Q(\Theta)_{iq}\Big[\left(\mathrm{diag}(1, \tfrac{1}{R}, 1)\mathbf{Q}(\Theta)^T\right)_{kj} Q(\Theta)_{js} \frac{\partial}{\partial \psi_k} T^{(R)}_{qs} \\ &\quad + \left(\mathrm{diag}(1, \tfrac{1}{R}, 1)\mathbf{Q}(\Theta)^T\right)_{2j} \Big(Q(\Theta)_{lq}(Q'(\Theta)_{ln}) T^{(R)}_{ns} Q(\Theta)_{js} + T^{(R)}_{qs}(Q'(\Theta)_{js})\Big)\Big] \\ &= Q(\Theta)_{iq}\Big[\mathrm{diag}(1, \tfrac{1}{R}, 1)_{ks} \frac{\partial}{\partial \psi_k} T^{(R)}_{qs} \\ &\quad + \mathrm{diag}(1, \tfrac{1}{R}, 1)_{2m} \Big(\mathbf{Q}(\Theta)^T \mathbf{Q}'(\Theta) \mathbf{T}^{(R)} + \mathbf{T}^{(R)} \mathbf{Q}'(\Theta)^T \mathbf{Q}(\Theta)\Big)_{qm}\Big],\end{aligned} \tag{A.43}$$

where

$$\mathbf{Q}'(\Theta) = \frac{\partial}{\partial\Theta}\mathbf{Q}(\Theta) = \begin{pmatrix} -\sin(\Theta) & -\cos(\Theta) & 0 \\ \cos(\Theta) & -\sin(\Theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$\mathbf{Q}(\Theta)^T\mathbf{Q}'(\Theta) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \mathbf{K}.$$

Therefore, by (A.43), the compressible equilibrium equations are equivalent to

$$\mathrm{diag}(1, \tfrac{1}{R}, 1)_{ks}\frac{\partial}{\partial\psi_k}T^{(R)}_{qs} + \mathrm{diag}(1, \tfrac{1}{R}, 1)_{2m}\left(\mathbf{K}\mathbf{T}^{(\mathbf{R})} + \mathbf{T}^{(\mathbf{R})}\mathbf{K}^{\mathbf{T}}\right)_{qm} = 0. \quad q = 1, 2, 3$$

This yields (A.40) by individually taking $q = 1, 2,$ and 3, and noting that $\mathbf{T}^{(R)}$ is symmetric. □

A.5 Results for Chapter 5

Lemma A.5.1. *Let* $a = \frac{77}{12} - \frac{2}{9}\log(2) > 0$, $b = -\frac{59}{36} + 4\log(2) > 0$, $c = -\frac{5}{3}$, *and* $d = \frac{4}{3}$. *Define the function* $f : (0, \infty)^2 \to \mathbb{R}$ *by*

$$f(r, L) := \left(aL + bL^3\right) + r^2\left(cL + \frac{d}{L}\right).$$

Then $f(r, L) < 0$ *if and only if* $L > \sqrt{\frac{4}{5}}$, *and*

$$r^2 > -\frac{b}{c}L^2 + \frac{bd - ac}{c^2} - \frac{bd^2 - ac^2}{cL^2 + d}. \tag{A.44}$$

Proof. We have that $f(r, L) < 0$ if and only if

$$aL + bL^3 + r^2\left(cL + \frac{d}{L}\right) < 0$$

$$\iff \quad \begin{cases} r^2 < -\frac{aL^2+bL^4}{cL^2+d}, & \text{if } cL^2 + d > 0 \\ aL + bL^3 < 0, & \text{if } cL^2 + d = 0 \\ r^2 > -\frac{aL^2+bL^4}{cL^2+d}, & \text{if } cL^2 + d < 0. \end{cases}$$

Checking each case individually,

<u>If $cL^2 + d > 0$:</u> Then $-\frac{aL^2+bL^4}{cL^2+d} < 0$, so we cannot have $f(r, L) < 0$.

If $cL^2 + d = 0$: Then $L^2 = -\frac{d}{c} = \frac{4}{5}$, so $aL + bL^3 > 0$. Therefore, we cannot have $f(r, L) < 0$.

If $cL^2 + d < 0$: In addition to $cL^2 + d < 0$ (which holds if and only if $L > \sqrt{\frac{4}{5}}$), we also require that $r^2 > -\frac{aL^2+bL^4}{cL^2+d}$, which is equivalent to (A.44).

$\square$

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